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THE MODES, RESONANCES AND FORCED RESPONSE OF ELASTIC STRUCTURES UNDER HEAVY FLUID LOADING

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CONTENTS

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This paper reports analytical studies of problems that involve the motion of plane elastic structures under conditions of heavy fluid loading. The main aspect concerns the description of the vibration response of a thin elastic plate (or membrane), of finite extent in at least one dimension, when the structure is excited by concentrated mechanical drive along a line or at a point; and as part of this the possibility of resonant response is discussed, and the resonance conditions and free modes of oscillation are obtained. There is also some discussion of the acoustic fields radiated by the structures under localized mechanical excitation.

The analysis makes extensive use of results for the reflection of a structural wave (subject to heavy fluid loading) at an edge, and the paper gives results for that reflection process covering waves incident normally on eight different edge configurations and waves incident obliquely on two edge configurations. These results include the reflection coefficient (whose magnitude is unity in the leading-order approximation of low-frequency heavy fluid loading), and the amplitude and directivity of the edge-scattered sound. By using the argument that edge reflection is a local process, the response is then calculated for a strip plate, under both line and point forcing, and the response is, for the first time, obtained for structures finite in both dimensions and subject to heavy fluid loading. Specifically, solutions are given here for a circular plate with eccentric drive, and for a membrane model of a rectangular panel, with central point drive. For some conditions and geometries expressions in simple form are found for the natural frequencies and mode shapes, and for the off-resonance forced response. Expressions for the drive admittances are found which display a variety of interesting features.

1. Introduction

1.1. *General remarks*

Vibrating elastic structures are to be found in many situations in Nature and in engineering. Only rarely do structures vibrate in a vacuum; commonly they are embedded in a fluid medium which may exert a significant pressure field, or *fluid loading*, on the structure. Fluid loading is a non-local mechanism; an attempt to deform a structure locally can lead to the generation of a slowly-decaying pressure field in the fluid and to consequent deformation of the structure

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further away. Fluid loading is *not* connected in any essential way with fluid compressibility. In some cases it may be important that the fluid motions producing the fluid loading have a phase and amplitude structure characteristic of acoustic perturbations; for example, the dissipation attributable to the radiation of acoustic energy to infinity may be a significant mechanism for limiting the vibration amplitude of a structural element near a resonance condition, while there are also many situations in which one wants to know the pattern of structural vibration produced in response to an incident acoustic field, or the acoustic field generated when the structure is driven by some mechanical excitation. In many other cases the length and time scales of interest are such that compressibility is of no significance.

Problems in which there is significant fluid loading are not easy to handle theoretically, even when the fluid has no mean motion and a very simple dynamic model and geometry are adopted for the structure. The problems may usually be treated as linear, but they have, in essence, the nature of wave diffraction problems in which all the familiar difficulties of diffraction theory are compounded by boundary conditions which couple, in a non-local way, surface waves on the structure to bulk waves in the fluid. As a satisfactory theoretical treatment of diffraction by a rigid strip has yet to be given, one can hardly expect a satisfactory treatment of the vibration of a finite fluid-loaded structure of any shape or mechanical properties. Considerable progress has, none the less, been made recently in the theory of fluid-loaded structures and it is the aim of the present paper to contribute to that progress with studies of finite geometries (circular and rectangular) under localized mechanical excitation and subject to 'heavy' fluid loading. That term ('heavy') is used with different implications by different authors, and we think it most important to define exactly what we mean by it and how our usage differs from that of others whose work appears partly to overlap with ours. After referring (in §1.2 below) to the appropriate prior work, we discuss the quantification of fluid loading (and introduce appropriate notation) in §1.3, relating this paper in §1.4 to our previous work.

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1.2. *Review of other work*

The simplest geometry, in which the structure is plane, homogeneous and infinite in extent, has been widely studied. Usually it is assumed that the same static inviscid fluid is present on both sides, or that there is a vacuum on one side, though it is formally easy to deal with different fluids. The structure may be idealized as a membrane, as a thin elastic plate or as a thick elastic plate (with the Timoshenko–Mindlin equation of motion), and the fields generated by a point source in the fluid or by a concentrated mechanical line force and moment or point-force excitation have been studied. Often the results are confined to the very distant structural and fluid wave fields or to the drive point behaviour, partly because these are quantities of physical interest and partly because there are efficient analytical techniques for obtaining them. More general results (covering all distances from an excitation and all frequencies) have been given by Crighton (1983), though that description is for the membrane model of the structure and is confined to the structural vibration field. The points mentioned above are covered in Morse & Ingard (1968, ch. 10), Junger & Feit (1972, ch. 7), Gutin (1965); Feit (1966); Nayak (1970); Crighton (1972, 1977, 1979) and Crighton & Innes (1983) and in numerous other papers.

The simplest inhomogeneities on a plane structure are those created by ribs and other supports (at points or along lines) on an otherwise infinite homogeneous structure. Formal solutions to problems involving such inhomogeneities can be obtained in terms of a fundamental

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(Green function) solution to the homogeneous problem. It is easy to deal thoroughly with the single inhomogeneity (Nayak 1970; Crighton & Maidanik 1981) though more interesting are multiple rib problems where the vacuum dynamics would lead to resonance possibilities. These have been examined by Leppington (1978), Stepanishen (1978) and others in the 'light' fluid loading limit (where one might expect the vacuum resonance conditions still to be important) and in 'heavy' fluid loading by Crighton & Innes (1983) (where, not so obviously, near-resonance can also occur under certain conditions). Periodic rib arrays have also been studied, both with regard to their diffraction grating behaviour (Leppington 1978; Stepanishen 1978; Konovalyuk 1969; Eatwell & Butler 1982; Mace 1980*a*, *b*) and with regard to their pass and stop band structure under heavy fluid loading, with implications for energy transfer down a periodically-ribbed structure under fluid loading (Crighton 1984*a*). The most recent studies in this direction concern the slightly randomly aperiodic array of ribs (Eatwell 1983; Crighton 1984*b*).

Inhomogeneities in the form of an edge (to a freely suspended or baffled plate) or abrupt thickness change need the solution of a Wiener–Hopf problem. This was first attempted by Lamb (1959), though his approximate factorization is valid (at best) only for light fluid loading. Davies (1974) gave an exact formal factorization and evaluated the relevant functions numerically to solve the problem of sound generation when a structural wave (subject to fluid loading) is incident on an edge of a membrane. The diffraction of a plane acoustic wave incident on the edge of a thin elastic plate was examined by Cannell (1975, 1976) for both light and heavy fluid loading (in a sense to be defined in §1.3) though for only one configuration and edge condition in each case. Cannell's results for heavy fluid loading were then adapted (with identical notation) by Abrahams (1981) who used them, together with matched asymptotic expansion methods, to analyse the plane wave diffraction by a large plate (of infinite length but finite width) set in a rigid baffle. The most striking result of this analysis was the prediction that the heavily fluid-loaded plate has *resonances* for particular widths and frequencies; the present writers know of no previous paper in which that possibility was even implied, though with hindsight it is perhaps not so surprising when one observes (as Abrahams did) that the waves incident on an edge are, in the heavy fluid loading limit, merely reflected with a phase change because of the essentially incompressible fluid motion which the edge reflection process generates. The waves at appropriate frequencies can therefore suffer multiple reflection and reverberant build-up until there is infinite amplitude in the steady state.

We have been aware of this argument for some time (since 1975, in fact) and the aim of the present paper is to exploit its application to a variety of geometries (including the rectangular and circular plate) and of edge constraints. Our idea of what constitutes 'heavy' fluid loading is, however, rather different from that of Cannell and Abrahams, as we shall now make clear.

1.3. *Parameters describing fluid loading*

To fix ideas and introduce notation, consider a thin elastic plate, of thickness h, Young modulus E and Poisson ratio ν, and of density ρ_p. Let the plate occupy the whole of $x_3 = 0$, with static fluid of density ρ and sound speed c_0 in $x_3 > 0$, a vacuum in $x_3 < 0$, and let the plate be driven by a line force $F_0\,\mathrm{e}^{-\mathrm{i}\omega t}$ per unit length of $0x_2$, acting in the $+x_3$-direction. The acoustic potential satisfies

$$(\nabla^2 + k_0^2)\,\phi = 0, \tag{1.1}$$

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with the kinematic condition

$$\partial\phi\,(x_1, 0)/\partial x_3 = -\mathrm{i}\omega\eta(x_1), \tag{1.2}$$

and the dynamic condition

$$B(\partial^4/\partial x_1^4 - k_p^4)\,\eta(x_1) = -\rho\mathrm{i}\omega\phi(x_1, 0) + F_0\,\delta(x_1). \tag{1.3}$$

A time factor $\exp(-\mathrm{i}\omega t)$, $\omega > 0$, is suppressed throughout. Other quantities introduced here are the bending stiffness, $B = Eh^3/12(1-\nu^2)$, the acoustic wavenumber $k_0 = \omega/c_0$, the vacuum bending wavenumber $k_p = (m\omega^2/B)^{\frac{1}{4}}$, the specific plate mass $m = \rho_p h$ and the plate deflection $\eta(x_1)$ in the x_3-direction, while $\delta(x_1)$ is the usual Dirac function. This problem contains (aside from an overall scaling factor proportional to F_0), two dimensionless parameters that specify the degree of fluid–structure coupling. We make the choice

$$\epsilon = \rho c_0/m\omega_g \quad \text{and} \quad M = k_0/k_p, \tag{1.4}$$

where ω_g is the so-called 'coincidence frequency', defined by $k_p(\omega_g) = k_0(\omega_g)$. M is a phase Mach number (ratio of wave speed on the plate in a vacuum to the sound speed) proportional to $\omega^{\frac{1}{2}}$, while ϵ is the ratio of fluid mass within a distance k_0^{-1} of the plate to the plate mass, evaluated at the coincidence frequency. Now, ϵ can be written as $\epsilon = (\rho/\rho_p)\,[E/12\rho_p c_0^2(1-\nu^2)]^{\frac{1}{2}}$, involving only the physical properties of the plate and fluid media and independent of both plate thickness and frequency. It thus provides an intrinsic measure of fluid loading, and will be called the *intrinsic fluid loading parameter*. Moreover, in most common applications ϵ is small; for steel in water $\epsilon = 0.133$, while for aluminium in air $\epsilon = 0.00213$.

Now the value of ϵ does not itself characterize fluid loading at any frequency ω; indeed the question of when fluid loading can be neglected (i.e. the term $-\rho\mathrm{i}\omega\phi$ neglected in (1.3)) does not have a unique answer, but depends on the magnitude of M and on the physical quantity under discussion. However, the choice of ϵ and M allows us to get round this difficulty; we regard ϵ as a small parameter and allow M to take all values. Use of matched expansion techniques then permits a systematic study of the whole frequency range and automatically preserves fluid-loading effects where they are needed. This programme was carried out by Crighton (1980) for the free wave and admittance properties of a fluid-loaded membrane, by Crighton & Maidanik (1981) for energy transmission across a rib on a fluid-loaded membrane, and by Crighton (1983) for the Green function (a function of ϵ, M and range x_0) of the fluid-loaded membrane. It emerges from those studies that, as far as the structural response goes, fluid-loading effects are light when $M = O(1)$, become significant when $M = O(\epsilon)$ and heavy when $M \ll \epsilon$. For example, the free surface wavenumber is $\kappa = k_p\{1+O(\epsilon)\}$ when $M = O(1)$, differs from k_p by an $O(1)$ factor when $M = O(\epsilon)$, and differs from k_p by a factor that tends to infinity when $M/\epsilon \to 0$; and we give the names light, significant and heavy to the degree of fluid loading in these three cases.

Heavy fluid loading is thus in our minds associated with the double limit

$$\epsilon \ll 1, \quad N = M/\epsilon \ll 1. \tag{1.5}$$

We envisage heavy fluid loading as achieved by taking a configuration with specified fluid and plate media (thus fixing ϵ at a small value) and progressively decreasing the frequency until the condition $N \ll 1$ is met.

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To see the way in which the parameters enter the model problem (1.1)–(1.3), introduce $\boldsymbol{x}^* = k_p \boldsymbol{x}$ and eliminate $\eta(x_1)$ to get

$$\left.\begin{aligned}(\nabla^{*2}+\epsilon^2 N^2)\,\phi &= 0,\\ [(\partial^4/\partial x_1^{*4}-1)\,\partial/\partial x_3^* + 1/N]\,\phi(x_1^*,0) &= -(\mathrm{i}\omega F_0/Bk_p^4)\,\delta(x_1^*).\end{aligned}\right\} \tag{1.6}$$

For $N = O(1)$ the only simplification is that the motion over length scales $O(k_p^{-1})$ is incompressible. For $N \ll 1$ a further simplification results if we define $\bar{\boldsymbol{x}} = N^{-\frac{1}{5}}\boldsymbol{x}^*$ and let $N \rightarrow 0$ with $\bar{\boldsymbol{x}}$ held fixed. Then at leading order the term corresponding to structural inertia drops out and we are left with a problem with no free parameters, namely

$$\left.\begin{aligned}\bar{\nabla}^2\phi &= 0,\\ [(\partial^4/\partial \bar{x}_1^4)\,(\partial/\partial \bar{x}_3)+1]\,\phi(\bar{x}_1,0) &= -(\mathrm{i}\omega F_0/Bk_p^4\,N^{\frac{1}{5}})\,\delta(\bar{x}_1).\end{aligned}\right\} \tag{1.7}$$

Here the mechanisms retained are those of structural stiffness, fluid inertia and (incompressible) pressure forces. The present paper is largely concerned with problems involving those three mechanisms, although on occasion we also consider (still with $\epsilon \rightarrow 0$, $N \rightarrow 0$) length scales of order k_0^{-1} for which the compressibility term in the Helmholtz equation must be retained.

For the most part it has not been necessary previously to go into much detail on what constitutes heavy fluid loading. Many authors, concerned with the distant acoustic field radiated from infinite homogeneous structures, have taken the condition for heavy fluid loading as $\rho c_0/m\omega \gg 1$, which corresponds to $M \ll \epsilon^{\frac{1}{2}}$ in our notation. This is correct for acoustic far-field problems, but not for the structural response, where one has to have $M = O(\epsilon)$ or less before there is a significant fluid-loading effect. Inhomogeneous (semi-infinite and finite) structures have, however, been discussed from the acoustic plane wave diffraction point of view by Cannell (1975, 1976) and Abrahams (1981) and it must be emphasized that their definition of heavy fluid loading is different from ours, with the consequence that any overlap that might have existed with the present work is very slight (and, to be specific, concerns just one entry of table 1 and the consequent prediction of certain resonance frequencies).

Cannell and Abrahams take as small a parameter which in our notation corresponds to $\epsilon N^{\frac{6}{5}}$. However, they also stipulate that as $\epsilon N^{\frac{6}{5}} \rightarrow 0$, the parameter M must remain $O(1)$. This may lead to a valid mathematical problem, but since it essentially sees heavy fluid loading as achieved by increasing the value of ρ at constant values of ω, the plate parameters and c_0, it does not correspond to the usual physical situation. There one has given media – steel and water, say – and is interested in whether fluid loading is light or heavy at a given frequency. When fluid loading is increased in this way by lowering the frequency it is simply not true that M can remain $O(1)$. A related difficulty is that the small parameter of Cannell and Abrahams is dependent on c_0, and would be zero in the incompressible limit, implying that in that limit fluid loading is necessarily heavy. That also is not true; the relevant measure of fluid loading for incompressible fluctuations is provided by the value of $N = mk_p/\rho$, and although we here take $N \ll 1$ there are cases of physical interest in which that is not appropriate.

We believe our choice of limiting processes to correspond better with the usual situations of interest and to provide the 'natural' way of quantifying fluid-loading effects. Certain results are, however, the same, whether our ordering is used, or that of Cannell and Abrahams. In particular, the phase change experienced by a wave on reflection from an edge should be the same (it is a local process and the relation between larger length scales is irrelevant), and we find that it is for the cases treated by Cannell and Abrahams. Other quantities need not be

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the same; for example, the acoustic field radiated in the edge reflection process is not locally determined, and we can see in the analysis that different terms are retained or neglected according as one takes $M = O(1)$ or $M \ll 1$. The differences are not qualitatively large, but the reader is cautioned none the less as to the possible differences which may arise from different hypotheses as to what amounts to heavy fluid loading.

1.4. *Scope of the present paper*

This paper must be related to a previous one (Crighton & Innes 1983). There the infinite plate, under line and point forcing, was examined in detail with the heavy fluid loading limit described above. Then *one* semi-infinite configuration was studied briefly, and with no proofs. The results for that configuration were then applied to the structural response of a strip plate of large but finite width, subject to line-force excitation. Resonant response was predicted at certain frequencies which agree with those found by Abrahams (1981), and other interesting aspects of the structural response were also delineated. Comparison was also made with the predictions of so-called 'edge-mode' theory, in which fluid loading is ignored in calculating the surface response which is then used to predict the acoustic field; the comparison showed edge-mode theory to be qualitatively and quantitatively wrong under heavy fluid loading.

We have several aims in this sequel paper. First, we wish to give a little detail and a great many results, most of them new, for the semi-infinite plate problem. These solutions, including for the first time problems of oblique incidence of a structural wave on an edge, are to serve as building blocks for finite geometries, and it is important that their properties be reasonably fully set down. Second, we wish to apply an asymptotic method, using the semi-infinite problem results, to finite geometries of interest, taking the strip with line force drive, the strip with point force drive, the circular plate with eccentric point drive and, finally, the rectangle with point drive (although only, in this first attempt, for the membrane model of the structure).

Under heavy fluid loading, the criteria by which these structures are to be regarded as large are simply (cf. the definition of $\boldsymbol{x}^*$ in §1.3)

$$\kappa l = k_p l N^{-\frac{1}{5}} \gg 1, \tag{1.8}$$

where l is a typical dimension. There is no requirement for the plate to be large on the acoustic scale (which would require $k_0 l = k_p l \epsilon N^{\frac{4}{5}} \gg 1$) and the possibilities $k_0 l \ll 1$ and $k_0 l \gg 1$ are both retained in the description of the acoustic field, where this is given. Our primary interest, however, is in the structural response provoked by localized mechanical forcing. Although our problems for finite geometry should really be attacked by matched asymptotic expansions, we do not use that formal language here. The problems are simple enough that it is not necessary to introduce the paraphernalia of multiple sets of dimensionless coordinates, and indeed most of the work is carried out directly in the dimensional coordinates.

2. Semi-infinite plates and membranes: the Wiener–Hopf method applied to unbaffled and baffled configurations

2.1. *Notation*

Throughout this paper the following conventions will be adopted. Static compressible fluid of density ρ and sound speed c_0 will occupy the region $x_3 > 0$ above a plane structure lying in $x_3 = 0$ if that structure is infinite (with a vacuum in $x_3 < 0$) and the whole space

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$-\infty < x_3 < \infty$ if the structure is not (doubly) infinite. (The reason for this is that when an infinite structure model is relevant, the case of most interest involves significant fluid loading on one side only; if double-sided fluid loading is of interest the necessary changes are easily made.) If the structure is an elastic plate it will have bending stiffness B, if a membrane the tension will be denoted by T. In either case, m will denote the mass per unit area and mechanical losses will be neglected. For two-dimensional problems, dependence on the x_2-coordinate will be ignored, the structure will lie along the x_1-axis and the fluid motion will take place in the (x_1, x_3) plane. For axisymmetric problems one defines $r = (x_1^2+x_2^2)^{\frac{1}{2}}$; other definitions to be used $k_\mathrm{p} = (m\omega^2/B)^{\frac{1}{4}}$, $k_\mathrm{m} = (m\omega^2/T)^{\frac{1}{2}}$ and $k_0 = \omega/c_0$ for the vacuum free wavenumbers at frequency ω on a plate or membrane, respectively, and for the acoustic wavenumber, while a fourth quantity of the same (wavenumber) dimensions is $\mu = \rho/m$. A time factor $\mathrm{e}^{-\mathrm{i}\omega t}$, $\omega > 0$, is suppressed throughout.

We begin by considering in some, though by no means full, detail a pair of prototypical two-dimensional problems involving semi-infinite unbaffled and baffled structures lying in equilibrium along $x_3 = 0$, $x_1 < 0$ and totally immersed in fluid so that the parameter μ must be interpreted as $2\rho/m$. The first of the two problems to be examined here has been briefly discussed by the authors (Crighton & Innes 1983), and it involves a coupled fluid – plate system which is excited by a subsonic surface wave normally incident from $x_1 = -\infty$ on the free edge of an unbaffled plate; in the second problem the structure lies adjacent to a rigid baffle. These two exemplify all the issues arising in a wide range of similar problems, and the aim is to sketch the procedures involved for those two and then merely to quote the corresponding results for six further cases.

2.2. *Free edge, unbaffled plate*

As is usual and necessary in problems of this type we express the total structural and acoustic fields as sums of incident and scattered parts, where the incident parts correspond merely to the bounded solution for the doubly-infinite system consisting of a free wave of wavenumber κ at frequency ω. Crighton (1979) has shown that, whatever the values of the physical parameters involved, the dispersion function

$$(\kappa^4 - k_\mathrm{p}^4)\,\gamma_\kappa - \mu k_\mathrm{p}^4 = 0 \tag{2.1}$$

always possesses precisely one positive real root; it is that wavenumber, representing an acoustically slow mode, which is designated henceforth by κ and in the heavy fluid loading limit it attains the value $(\mu k_\mathrm{p}^4)^{\frac{1}{5}}$. Furthermore, it is evident that the scattered acoustic field is odd in x_3 and thus our attention is confined to $x_3 \geqslant 0$. Then, the definitive equations for the unbaffled system comprise the Helmholtz equation

$$(\nabla^2 + k_0^2)\,\phi = 0 \tag{2.2}$$

for the fluid potential, the equation for the plate deflection

$$B(\partial^4/\partial x_1^4 - k_\mathrm{p}^4)\,\eta = -2\rho\mathrm{i}\omega\phi(x_1 < 0,\, x_3 = 0_+), \tag{2.3}$$

the kinematic condition

$$\partial\phi(x_1, 0)/\partial x_3 = -\mathrm{i}\omega\eta(x_1), \quad x_1 < 0, \tag{2.4}$$

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and the symmetry condition for the unbaffled configuration

$$\phi(x_1, 0) = -(\mathrm{i}\omega d/\gamma_\kappa) \exp(\mathrm{i}\kappa x_1), \quad x_1 > 0, \tag{2.5}$$

where ϕ and η are the potential and deflection in the scattered wave and $d \exp(\mathrm{i}\kappa x_1)$ is the plate deflection in the incident wave.

To analyse the problem we introduce the generic Fourier transforms (cf. Noble 1958) defined by

$$\tilde{f}_-, \tilde{f}_+(k, x_3) = \frac{1}{2\pi} \int_{-\infty}^{0}, \int_{0}^{+\infty} f(x_1, x_3) \exp(\mathrm{i}kx_1)\, \mathrm{d}x_1. \tag{2.6}$$

At the outset it is necessary to introduce a small positive amount of dissipation into the system by attributing a small positive imaginary part to the frequency. This device ensures that the half-range Fourier transforms of the scattered fields exist as analytic functions of k in some common strip defined by the intersection of overlapping upper and lower half-planes $R_\pm$ and that functions suffixed by $\pm$ possess the usual analyticity properties in the regions $R_\pm$ of the complex wavenumber plane.

It is then a simple matter to manipulate the transformed equations and boundary conditions to obtain a linear functional equation of the Wiener–Hopf type, namely

$$K(k)\,[\tilde{\phi}_-(k, 0) + \omega d/\gamma_\kappa(k+\kappa)] + [(k^4 - k_\mathrm{p}^4)\, \partial\tilde{\phi}_+(k, 0)/\partial x_3 + \mathrm{i}\omega P(k)] + (2\rho\omega^2/B)\, \omega d/\gamma_\kappa(k+\kappa) = 0, \quad k \in R_+ \cap R_-, \tag{2.7}$$

wherein the kernel, $K(k)$, is that ubiquitous function of coupled fluid–plate problems given by

$$K(k) = (k^4 - k_\mathrm{p}^4)\, \gamma - \mu k_\mathrm{p}^4 \tag{2.8}$$

and

$$\gamma = (k^2 - k_0^2)^{\frac{1}{2}}, \tag{2.9}$$

with branch cuts defined in the usual manner, that is running from the branch points to infinity in the upper and lower half-planes without crossing the strip of analyticity. The polynomial

$$P(k) = [\partial^3\eta(0)/\partial x_1^3 + (-\mathrm{i}k)\, \partial^2\eta(0)/\partial x_1^2 + (-\mathrm{i}k)^2\, \partial\eta(0)/\partial x_1 + (-\mathrm{i}k)^3\, \eta(0)]$$

is related to the edge constraints; in it we regard any two of the coefficients as being prescribed by those constraints and the remaining two as unknown constants (to be determined as part of the solution). In particular, at a *free* edge the total force and the bending moment are both zero, from which it follows that

$$\partial^3\eta(0)/\partial x_1^3 = \mathrm{i}\kappa^3 d, \quad \text{and} \quad \partial^2\eta(0)/\partial x_1^2 = \kappa^2 d. \tag{2.10}$$

Splitting the *even* function $K(k)$ into a product of factors $K_+(k)\, K_-(k)$, analytic and non-zero in $R_\pm$ respectively, and with

$$K_+(-k) = K_-(k), \tag{2.11}$$

and

$$K_\pm(k) = O(k^{\frac{5}{2}}), \tag{2.12}$$

as $|k| \to \infty$ in $R_\pm$, the standard procedure (Noble 1958, p. 36ff) leads to a pair of algebraic equations involving an entire polynomial which arises naturally via the application of the extended form of Liouville's theorem following the Wiener–Hopf split manipulation. However,

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the familiar anticipated behaviour at the edge, that the pressure be bounded and the fluid velocity be no more singular than $|\boldsymbol{x}|^{-\frac{1}{2}}$, allows us to deduce that in the present problem

$$E(k) = E_0 + E_1 k, \tag{2.13}$$

thereby introducing a further two unknown constants. So finally we have the solution to the Wiener–Hopf equation (2.7) in the form

$$\tilde{\phi}_-(k,0) = (E_0+E_1 k)/K_-(k) - [\omega d/\gamma_\kappa(k+\kappa)]\,[1-K_-(-\kappa)/K_-(k)], \tag{2.14}$$

and

$$(k^4-k_p^4)\,\partial\tilde{\phi}_+(k,0)/\partial x_3 = -(E_0+E_1 k)\,K_+(k) - [\omega d/\gamma_\kappa(k+\kappa)] \times [2\rho\omega^2/B + K_-(-\kappa)\,K_+(k)] - \mathrm{i}\omega P(k). \tag{2.15}$$

In this unbaffled configuration the four unknown constants are determined in a familiar manner by ensuring that all subscripted functions are analytic throughout their domains of analyticity. Specifically, we note that $\partial\phi_+/\partial x_3$ appears to possess poles at $k = +k_p$ and $+\mathrm{i}k_p$. Therefore we choose the constants so that these poles are in fact illusory and this requires that

$$K_+(\lambda)\,(E_0+E_1\lambda) + [2\rho\omega^2/B + K_-(-\kappa)\,K_+(\lambda)]\,[\omega d/\gamma_\kappa(k+\kappa)] + \mathrm{i}\omega P(\lambda) = 0, \tag{2.16}$$

with $\lambda = +k_p$ and $+\mathrm{i}k_p$ in turn. Similarly, an inspection of

$$\tilde{\eta}_-(k) = -[(2\rho\mathrm{i}\omega/B)\,\tilde{\phi}_-(k,0) + BP(k)]/(k^4-k_p^4) \tag{2.17}$$

yields two additional equations:

$$\frac{(E_0+E_1\lambda)}{K_-(\lambda)} - \frac{\omega d}{\gamma_\kappa(k+\kappa)}\,[1-K_-(-\kappa)/K_-(\lambda)] + \frac{B}{2\rho\mathrm{i}\omega}P(\lambda) = 0, \tag{2.18}$$

for $\lambda = -k_p$ and $-\mathrm{i}k_p$.

The set of equations, (2.16) and (2.18), is sufficient to determine the four unknown constants and thus the scattered field, though not in a guise which is particularly illuminating with regard to the prediction of characteristics of the diffracted and reflected waves. None the less, in the low frequency–heavy fluid loading limit the explicit expressions of the Appendix ((A 17) and (A 22)) for the factors $K_\pm(k)$ may be used to obtain transparently simple results in a number of interesting cases.

Specifically, when the edge at $x_1 = 0$ is free and in the heavy fluid loading limit, we obtain

$$E_0 \sim \omega d k_p^{\frac{1}{2}}\,10^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{8}\pi\mathrm{i}}/N^{\frac{1}{10}}, \quad \text{and} \quad E_1 \sim -\omega d\,10^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{8}\pi\mathrm{i}}\,N^{\frac{1}{10}}/k_p^{\frac{1}{2}}, \tag{2.19}$$

(where the definitions $\mu/k_p = \epsilon/M$, $k_0/k_p = M$, $M = \epsilon N$ have been introduced and in the low frequency–heavy fluid loading limit $N \ll 1$).

For the plate waves reflected and scattered at the edge we have

$$\eta(x_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\eta}(-k)\exp(\mathrm{i}kx_1)\,\mathrm{d}k, \tag{2.20}$$

in which $\tilde{\eta}(k)$ is given by a combination of equations (2.14) and (2.17). In the limit of vanishing dissipation the path of integration becomes the real axis indented above (below) poles and branch points on its negative (positive) ranges. For $x_1 < 0$ we deform this contour to infinity

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in R_-. It is a simple matter to verify that the only singularities of $\tilde{\eta}$ are those associated with the branch point and poles of $K_+(k) = K(k)/K_-(k)$. Complex poles in R_- necessarily lead to exponentially decaying coupled modes; and branch line integral contributions decay algebraically, as $O(|x_1|^{-\frac{3}{2}})$, when $x_1 \to \infty$ on the plate. Thus the only propagating plate wave arises from the residue contribution at $k = -\kappa$ and this leads to a reflected wave

$$\eta_r(x_1) = \frac{2\rho\omega}{B} \frac{K_-(-\kappa)}{[\mathrm{d}K(k)/\mathrm{d}k]_{k=\kappa}} \left[E_0 + E_1\kappa + \frac{\omega d K_-(-\kappa)}{2\kappa\gamma_\kappa}\right] \frac{\mathrm{e}^{-\mathrm{i}\kappa x_1}}{(\kappa^4 - k_p^4)}, \tag{2.21}$$

which, in the limit under consideration, reduces to

$$\eta_r(x_1) = d\,\mathrm{e}^{-\mathrm{i}(\kappa x_1 - \frac{1}{4}\pi)}, \tag{2.22}$$

demonstrating, as anticipated, that, in a first approximation, the incident field merely suffers a phase change on reflection without change in amplitude.

The distant field may be calculated from the inverse of the transformed Helmholtz equation:

$$\phi(x_1, x_3) = \frac{\operatorname{sgn} x_3}{2\pi} \int_{-\infty}^{+\infty} \tilde{\phi}(k, 0) \exp(-\mathrm{i}k_1 x_1 - \gamma|x_3|)\,\mathrm{d}k \tag{2.23}$$

with integrand given, in the unbaffled case, by

$$\tilde{\phi}(k, 0) = [E_0 + E_1 k + \omega d K_-(-\kappa)/\gamma_\kappa(k+\kappa)]/K_-(k). \tag{2.24}$$

The path of integration is from $-\infty$ to $+\infty$ within the strip of analyticity and this shrinks to the indebted real axis as before. Again we omit details of the standard contour deformation onto the steepest descents path (Noble 1958; Clemmow 1966) and we simply quote the familiar result that the distant radiating acoustic field may be evaluated from

$$\phi(x_1, x_3) \sim \left(\frac{k_0}{2\pi r_*}\right)^{\frac{1}{2}} \exp(\mathrm{i}k_0 r_* - \tfrac{1}{4}\mathrm{i}\pi) \sin\theta\, \tilde{\phi}(-k_0\cos\theta, 0) \operatorname{sgn} x_3, \tag{2.25}$$

where $x_3 = r_* \sin\theta$, $r = r_* \cos\theta$. When use is made of the leading-order terms alone for E_0 and E_1 it becomes obvious that the 'natural' leading-order term in $\tilde{\phi}(-k_0\cos\theta, 0)$ vanishes, and therefore it is necessary to retain higher-order terms in the various asymptotic expansions required in the determination of the four unknown constants. The simplest way of dealing with the four equations in (2.16) and (2.18) is to rewrite them as indicated in the Appendix ((A 46)–(A 50)). From the alternative set of equations given there it is a straightforward, though tedious, matter to obtain values for the constants which lead to the following expression for the diffracted field:

$$\Phi_d(x_1, x_3) \sim [c_0 d \exp(\mathrm{i}k_0 r_*)/(\pi k_0 r_*)^{\frac{1}{2}}] \sin\theta\, q\epsilon^2 N^{\frac{16}{5}} \operatorname{sgn} x_3, \tag{2.26}$$

where the *number* q is $(\frac{1}{5})^{\frac{1}{2}} \mathrm{e}^{-\frac{5}{8}\pi\mathrm{i}}/\sin\frac{1}{10}\pi$. This completes our examination of the semi-infinite unbaffled plate for normally incident structural waves.

2.3. *Free edge, baffled plate*

To continue our study of semi-infinite wavebearing surfaces with normal incidence at the edge, we consider briefly how the foregoing discussion should be amended when the plate is abutted to a perfectly rigid baffle occupying the half-plane $x_3 = 0$, $x_1 > 0$. The edge is not constrained at the junction and is able to move freely next to the baffle.

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The presence of the baffle changes the condition on $x_3 = 0$, $x_1 > 0$ and (2.5) is replaced by the kinematic condition

$$\partial\phi(x_1, 0)/\partial x_3 = \mathrm{i}\omega d \exp(\mathrm{i}\kappa x_1), \quad x_1 > 0. \tag{2.27}$$

Now we proceed as before to a Wiener–Hopf equation

$$H(k)\,[\tilde{\eta}_-(k) - \mathrm{i}d/(k+\kappa)] + \mathrm{i}d(k^4 - k_\mathrm{p}^4)/(k+\kappa) + P(k) - (2\rho\mathrm{i}\omega/B)\,\tilde{\phi}_+(k,0) = 0, \quad k \in R_+ \cap R_-, \tag{2.28}$$

with the kernel, $H(k)$, now given by

$$H(k) = K(k)/\gamma. \tag{2.29}$$

This has a solution

$$\tilde{\eta}_-(k) = (E_0 + E_1 k)/H_-(k) + [\mathrm{i}d/(k+\kappa)]\,[1 - H_-(-\kappa)/H_-(k)], \tag{2.30}$$

and
$$\tilde{\phi}_+(k,0) = \frac{B}{2\rho\mathrm{i}\omega}\left\{\left[(E_0 + E_1 k) - \frac{\mathrm{i}d}{(k+\kappa)} H_-(-\kappa)\right] H_+(k) - \frac{\mathrm{i}d(k^4 - k_\mathrm{p}^4)}{(k+\kappa)} + P(k)\right\}. \tag{2.31}$$

In this case the most convenient way of determining the entire function is by examination of the asymptotic form of (2.30) as $|k| \to \infty$ in R_-. By using the results of the Appendix, which are pertinent in the heavy fluid-loading limit, the functions on the right side of the equation for $\tilde{\eta}_-(k)$ may be expanded in inverse powers of k up to the term in k^{-4}. Thence Watson's lemma (Murray 1974) may be invoked in an inverse fashion to find the behaviour of $\eta(x_1)$ as $x_1 \to 0_-$. The upshot of this procedure is that

$$\begin{aligned}\eta(x_1) \sim\ & \mathrm{i}(E_0/h_0 + \mathrm{i}d) + (E_0/h_0 - E_1 h_1/h_0 - \mathrm{i}\kappa d)\, x_1 \\ & - \mathrm{i}[-E_1 h_1/h_0 + (E_1/h_0)\,(h_1^2 - h_2) + \mathrm{i}\kappa^2 d - (\mathrm{i}d/h_0)\,H_-(-\kappa)]\,\tfrac{1}{2}x_1^2 \\ & - [(E_0/h_0)\,(h_1^2 - h_2) + (E_1/h_0)\,(2h_1 h_2 - h_3 - h_1^3) \\ & - \mathrm{i}\kappa^3 d + (\mathrm{i}d/h_0)\,H_-(-\kappa)\,(\kappa + h_1)]\,\tfrac{1}{6}x_1^3 + O(x_1^4), \quad \text{as} \quad x_1 \to 0_-,\end{aligned} \tag{2.32}$$

where, for heavy fluid loading, the trigonometric expressions h_i $(i = 0, 1, 2, 3)$ are given in the Appendix (A 44). Equation (2.32) is exact and not specific to any given edge constraint. However, in the particular case of a free edge, when (2.10) is valid, the unknown constants can be determined exactly from (2.32) and their values are also given in the Appendix (A 50).

Arguments identical to those previously used show that the reflected elastic plate wave is of the form

$$\eta_\mathrm{r}(x_1) = -\frac{d[H_-(-\kappa)]^2}{2\kappa\{\mathrm{d}[H(k)]/\mathrm{d}k\}_{k=-\kappa}}\left\{1 + \frac{2\kappa(\kappa^2 h_1 + \kappa h_1^2 + h_1 h_2 - h_3)}{(h_2^2 - h_1 h_3)}\right\}\mathrm{e}^{-\mathrm{i}\kappa x_1}, \tag{2.33}$$

and this simplifies to
$$\eta_\mathrm{r}(x_1) \sim d\exp[-\mathrm{i}(\kappa x_1 - \tfrac{7}{20}\pi)], \tag{2.34}$$

use having been made of the explicit values of the h_i in the heavy fluid loading limit.

Equation (2.25) still provides an asymptotic formula for evaluating the distant acoustic field. However, in this case,

$$\tilde{\phi}(k,0) = (\mathrm{i}\omega/\gamma)\,[E_0 + E_1 k - \mathrm{i}dH_-(-\kappa)/(k+\kappa)]/H_-(k), \tag{2.35}$$

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and for heavy fluid loading this leads directly to

$$\Phi_d \sim (c_0 d)\,[\exp(ik_0 r_*)/(\pi k_0 r_*)^{\frac{1}{2}}]\cos(\tfrac{1}{2}\theta)\, q\epsilon^{\frac{3}{2}} N^{\frac{9}{5}} \tag{2.36}$$

where, here, $$q = 10^{\frac{1}{2}}\sin(\tfrac{1}{10}\pi)\,e^{\frac{27}{40}\pi i}.$$

2.4. *Discussion*

The principal result of the section is that, for a free edge, an incident plate wave $d\exp(i\kappa x_1)$ is reflected as $Rd\exp(-i\kappa x_1)$ together with, of course, an acoustic field which decays algebraically as $|x_1| \to \infty$ on the plate. R has the value $\exp(+\frac{7}{20}\pi i)$ or $\exp(\frac{1}{4}i\pi)$ according to whether the plate is baffled or not. In either case R is equivalent merely to a phase shift and this implies that, to leading order, when $N \to 0$, no energy is lost in the reflection process. Indeed this can also be shown to be true in the régime of more modest fluid loading characterized by $N = O(1)$, but there the phase shift is a complicated function of frequency which simplifies to the frequency independent values of $+\frac{7}{20}\pi$ and $\frac{1}{4}\pi$, respectively, as $N \to 0$. As for the scattered field, we have shown that it varies as $N^{\frac{9}{5}}\cos\frac{1}{2}\theta$ and $N^{\frac{16}{5}}\sin\theta$ in the two cases under consideration. A comparison of the results, for the unbaffled case, with those predicted by the familiar ideas of 'edge-mode' radiation was included in Crighton & Innes (1983); there it was shown that those ideas greatly overestimate the strength of the scattered field, while in circumstances of light fluid loading it has been shown (Cannell 1975) that the predictions of 'edge-mode' theory are accurate.

However, the arguments and equations of this section are, for the most part, general and not special to the free edge condition (nor in fact to the heavy fluid-loading limit though attention will be restricted to that limit here). Thus it is possible to use these two distinct methods for unbaffled and baffled plates to predict, in the very heavy fluid-loading limit, the phase shift on reflection and the level and directivity of the diffracted field for a whole variety of alternative edge constraints; this we have done and the results are tabulated and discussed in the following section.

3. Table of results for directivity and phase shift for various configurations under heavy fluid loading

We have used the equations of the preceding section to evaluate the reflection coefficient and the acoustic field for eight different semi-infinite configurations. Six of these involve a plate baffled by an adjacent semi-infinite rigid plate, or unbaffled, and with three different edge conditions. The two remaining cases involve an unbaffled membrane with either a free or fixed edge; these are included for future reference when further extensions to finite strips and panels involving other types of forcing are considered.

The results for the phase shift Θ, defined by $R = \exp(i\Theta)$, and for the far field dependence on N, ϵ and θ are recorded in table 1. This reveals a wide, though not simply predicted, variation in the reflection coefficient. The strongest radiated acoustic fields are those associated with baffled geometries and typically these are stronger by a factor $O(N^{-\frac{4}{5}})$ in amplitude than those arising in the unbaffled configurations. The weaker fields scattered by an unbaffled edge have two different variations with N, the weakest of these fields being that associated with the free edge. We also note that the application of the Wiener–Hopf technique, which is necessary to

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Table 1. The phase shift Θ (defined by $R = \exp(\mathrm{i}\Theta)$) and the variation of the amplitude of the scattered acoustic field with N, ϵ and θ, for six different semi-infinite plate and two semi-infinite membrane configurations

(The function F and the number q are defined so that, in all cases, $\Phi_{\mathrm{d}} = (1/\pi k_0 r_*)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}k_0 r_*} (c_0 d)\, qF(\epsilon, N, \theta)$.)

	edge condition	phase shift, Θ	acoustic field $F(\epsilon, N, \theta)$	acoustic field q
unbaffled plates	freely suspended, free edge	$\frac{1}{4}\pi$	$\epsilon^2 N^{\frac{16}{5}} \sin\theta$	$(\frac{1}{5})^{\frac{1}{2}} \mathrm{e}^{-\frac{5}{8}\pi \mathrm{i}}/\sin(\frac{1}{10}\pi)$
	clamped edge	$-\frac{3}{20}\pi$	$\epsilon^2 N^{\frac{13}{5}} \sin\theta$	$(\frac{5}{2})^{\frac{1}{2}} \tan(\frac{1}{5}\pi)\, \mathrm{e}^{-\frac{11}{40}\pi \mathrm{i}}$
	simple support	$\frac{1}{4}\pi$	$\epsilon^2 N^{\frac{13}{5}} \sin\theta$	$(10)^{\frac{1}{2}} \sin(\frac{1}{5}\pi)\, \mathrm{e}^{-\frac{3}{40}\pi \mathrm{i}}$
baffled plates	free edge	$\frac{7}{20}\pi$	$\epsilon^{\frac{3}{2}} N^{\frac{9}{5}} \cos\frac{1}{2}\theta$	$(10)^{\frac{1}{2}} \sin(\frac{1}{10}\pi)\, \mathrm{e}^{\frac{27}{40}\pi \mathrm{i}}$
	clamped edge	$\frac{3}{4}\pi$	$\epsilon^{\frac{3}{2}} N^{\frac{9}{5}} \cos\frac{1}{2}\theta$	$(10)^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{8}\pi \mathrm{i}}$
	pin jointed	$\frac{23}{20}\pi$	$\epsilon^{\frac{3}{2}} N^{\frac{9}{5}} \cos\frac{1}{2}\theta$	$(10)^{\frac{1}{2}} \cos(\frac{1}{5}\pi)\, \mathrm{e}^{\frac{3}{40}\pi \mathrm{i}}$
unbaffled membranes	free edge	$-\frac{1}{4}\pi$	$\epsilon^2 N^{\frac{10}{3}} \sin\theta$	$\mathrm{e}^{-\frac{7}{8}\pi \mathrm{i}}$
	fixed edge	$-\frac{11}{12}\pi$	$\epsilon^2 N^{\frac{8}{3}} \sin\theta$	$\frac{3}{2} \mathrm{e}^{-\frac{5}{24}\pi \mathrm{i}}$

implement the ideas of edge-mode theory, in the unbaffled cases leads to a multipole $\sin\frac{1}{2}\theta$ directivity in contrast to the dipole directivity $\sin\theta$ of our exact solution; likewise, edge-mode theory for baffled plates, which requires no Wiener–Hopf analysis, predicts dipole directivity whereas our solution now varies as $\cos\frac{1}{2}\theta$.

4. The energy balance in the coupled fluid–plate system

4.1. *Energy fluxes in the plate*

It is of some interest to consider the balance of power in the coupled fluid–plate system both as a partial check on the previous calculations and to see how the incident wavepower is distributed among the available modes. An energy equation for the system is readily obtained from the time-dependent linear structural and acoustic equations in a familiar way. Such an equation was derived for the membrane by Crighton (1984*c*) and the corresponding results for the plate are easy to derive. They lead to

$$2\left\langle \int_{\mathrm{S}} p \frac{\partial \phi}{\partial n} \mathrm{dS} \right\rangle = \langle F(\bar{x}_1) - F(\bar{x}_2) \rangle, \tag{4.1}$$

with the time average $\langle\ \rangle$ calculated from

$$\langle f_1(x_1, t) f_2(x_1, t) \rangle = \tfrac{1}{2} \mathrm{Re}\{f_1(x_1) f_2^*(x_1)\}$$

and where * denotes complex conjugation. Here, S is a surface in the fluid joining the points $\bar{x}_1$, $\bar{x}_2$ on the plate, $\boldsymbol{n}$ is the unit outward normal, and $F(x_1)$ is the mechanical flux across station x_1 in the plate, given by

$$F(x_1) = B\left[\frac{\partial^3 \eta}{\partial x_1^3} \frac{\partial \eta}{\partial t} - \frac{\partial^2 \eta}{\partial x_1^2} \frac{\partial^2 \eta}{\partial x_1 \partial t}\right]. \tag{4.2}$$

We now consider how the power balance expressed by (4.1) is achieved in one of the typical cases previously discussed in §2. To calculate the mechanical flux $\langle F(x_1) \rangle$ in the plate itself we note that the total structural field comprises propagating incident and reflected waves,

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non-propagating waves associated with complex poles in R_+ of $K_-(k)$, and an acoustic field allied to the branch line integral. These last two contributions to the field both decay as $|x_1|\to\infty$ on the plate, the first necessarily exponentially and the second as $O(|x_1|)^{-\frac{3}{2}}$; thus they transport no energy to infinity along the plate. Consequently the mechanical energy flux is all carried by the subsonic surface waves. Now at great distances from the edge we may write

$$\eta = d(e^{i\kappa x_1}+R\,e^{-i\kappa x_1})\,e^{-i\omega t}, \tag{4.3}$$

in which R is the complex number depending crucially on the actual configuration under consideration, but which *always* reduces, in the very heavy fluid-loading limit $N\to 0$, to $R = e^{i\Theta}(1+O(M^{\frac{4}{5}}))$. (Here Θ is the phase shift of table 1.) It follows (cf. Innes 1983, p. 87) that the net mechanical energy flux in the positive x_1-direction, across a distant section x_1 of the plate, can be determined in terms of the total structural field (4.3) by

$$\langle F(x_1)\rangle = B\omega d^2\kappa^3(1-|R|^2), \tag{4.4}$$

and, of this total, a quantity $B\omega d^2\kappa^3$ is carried by the incident wave. To calculate the rate of working of the pressure across S, $\langle\int_S p(\partial\phi/\partial n)\,dS\rangle$, we recall that the far field in the fluid has two distinct components that are capable of carrying energy: first, a wavefield thrown up when the subsonic wavepole $k=\kappa$ is captured during the deformation of the integration path on to a steepest descents path and second, a cylindrical acoustic wave of general form $r_*^{-\frac{1}{2}}\exp(ik_0 r_*)\,\Phi(\theta)$ which arises from the steepest descents integral alone. It is easily shown that, when the surface S is sufficiently large, the total rate of working of the pressures across S is simply the sum of the individual rates associated with the two pressure fields separately. Now the field induced in the fluid by the motion of the plate is given by

$$\phi = (i\omega d/\gamma_\kappa)\,[e^{i\kappa x_1}+R\,e^{-i\kappa x_1}]\,e^{-\gamma_\kappa x_3}\,e^{-i\omega t}, \quad x_3>0, \tag{4.5}$$

and we see that although this field decays exponentially with normal distance from the plate x_3, it does indeed carry energy to infinity in planes parallel to the plate. Thus the most appropriate choice of S for this field is a large prism of square cross section with sides parallel to the x_2 and x_3 axes. Thence, for the subsonic nearfield pressure wave, we have

$$\left\langle\int_S p\frac{\partial\phi}{\partial n}\,dS\right\rangle = 2\left\langle\int_0^\infty \rho i\omega\phi\frac{\partial\phi}{\partial x_1}\,dx_3\right\rangle = (\rho\omega^3 d^2\kappa/2\gamma_\kappa^3)\,(1-|R|^2), \tag{4.6}$$

of which the amount $\rho\omega^3 d^2\kappa/2\gamma_\kappa^3$ is carried by the incoming wave. The total average input energy flux is therefore the sum of the first terms in (4.4) and (4.6), namely

$$\begin{aligned}\langle F(x_1)\rangle_{\text{input}} &= B\omega d^2\kappa^3+\rho\omega^3 d^2\kappa/2\gamma_\kappa^3\\ &= (m\omega^3 d^2\kappa^3/k_p^4)\,\{1+\tfrac{1}{4}[(\kappa/k_p)^4-1]^3/(\kappa/k_p)^2\,(\epsilon/M)^2\}.\end{aligned}$$

Now for $M=O(1)$, $\epsilon\ll 1$

$$\langle F(x_1)\rangle_{\text{input}} \sim (m\omega^3 d^2/k_p)\,\{1+\epsilon(4-3M^2)/4M(1-M^2)^{\frac{3}{2}}\}, \tag{4.7}$$

but in the very heavy fluid-loading limit, $M=\epsilon N$, $\epsilon\ll 1$,

$$\langle F(x_1)\rangle_{\text{input}} \sim (m\omega^3 d^2/k_p N^{\frac{3}{5}})\,(1+\tfrac{1}{4}). \tag{4.8}$$

Thus, while in the light fluid-loading limit an asymptotically negligible fraction of the incident energy resides in the fluid, we see that in the heavy fluid-loading limit the situation differs

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significantly: of the incident energy $\frac{4}{5}$ resides in the plate and $\frac{1}{5}$ in the induced subsonic pressure wave, which is analogous to the result previously obtained (Crighton 1984*c*) for the membrane.

Finally we come to the radiating acoustic field scattered by the edge at $x_1 = 0$. To evaluate the associated flux we choose S to be a large cylinder, of obvious orientation, so that for the cylindrical wave

$$\left\langle \int_S p \frac{\partial \phi}{\partial n} \mathrm{d}S \right\rangle = \rho_0 k_0 \omega \int_0^\pi |\Phi(\theta)|^2 \, \mathrm{d}\theta \tag{4.9}$$

for double-sided fluid loading. For the two specific cases considered in some detail in §2 we find that the acoustic energy fluxes per unit length parallel to the edge carried by the cylindrical wave, are

$$m\omega^3 d^2 \epsilon^2 N^{\frac{17}{5}}/k_\mathrm{p}\, 20 \sin^2 \tfrac{1}{10}\pi \quad \text{and} \quad (m\omega^3 d^2 \epsilon N^{\frac{3}{5}}/k_\mathrm{p})\, \tfrac{5}{2} \sin^2 \tfrac{1}{10}\pi \tag{4.10}$$

for the unbaffled and baffled cases respectively. Thus, even in the very heavy fluid-loading limit, the acoustic energy flux in the cylindrical wave is a negligible fraction of the incident wavepower. Furthermore, in this heavy fluid-loading limit it has been shown that, for all the configurations tabulated in §3, $|R| = 1$ to leading order. This is simply a manifestation of the fact that in this limit the fluid is virtually incompressible (as inferred from the replacement of γ by $|k|$ to leading order, for example). Hence it follows from (4.4) that, to leading order in the assumed limit, the total reflected wavepower, carried in the surface itself and in the adjacent fluid layer of thickness $2\gamma_\kappa^{-1}$, precisely balances the incoming energy flux.

It is, of course, possible to calculate subsequent terms in the expression for the reflection coefficient, R, and this we have done. But what is needed in any further consideration of the edge-scattering process is, typically, several terms of uniformly valid asymptotic expressions for the Wiener–Hopf kernel factor $K_+(k)$ when $k = O(1)$, $O(N^{-\frac{1}{5}})$ and as $|k| \to \infty$; moreover, until a stage is reached when it is necessary to retain the term in the kernel which is identified with *fluid compressibility* the corrections to the reflection coefficient obtained *contribute adjustments to the phase shift alone*. Although the generation of the required asymptotic series is a straightforward matter, an inspection of the (re-scaled) kernel function in the form

$$K(t) = (t^4 - M^{\frac{4}{5}}/\sigma^4)\,(t^2 - M^{\frac{12}{5}}/\sigma^2)^{\frac{1}{2}} - 1 \quad \text{(see (A 4) with } z = t\sigma/M^{\frac{1}{5}}\text{)},$$

makes it obvious why *many* terms of the series are required before the effects of fluid compressibility come into the reckoning; thus the whole business of determining the unknown constants from, say (2.16) and (2.18) becomes extremely arduous and time-consuming. However, the roles played by the individual mechanisms involved in the reflection and scattering processes at the edge, particularly that of fluid compressibility may be exposed in a far more convincing manner. The original thin plate–fluid system will be discarded and will be replaced by one in which the structural component is redefined in 'lumped circuit' terms chosen, crucially, to highlight those physical attributes which have been revealed as dominant in the coupled plate–fluid complex and the low frequency limit.

4.2. *Complete power balance for a locally reacting surface*

Accordingly, a simpler semi-infinite model is considered. The thin elastic plate is replaced by a simple damped mass–spring system for which the local surface impedance Z is given in terms of its resistive and reactive components, R and X respectively, by

$$Z = -R - \mathrm{i}X, \tag{4.11}$$

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wherein *both* R and X are *positive* constants. This type of approach has proved useful before in modelling simple coupled systems where the adoption of a more complicated non-local dynamical equation for the surface response fails to admit sufficient tractibility (see, for example, Crighton & Leppington 1970). Now, in the low frequency-very heavy fluid-loading limit which is of interest here, it is known that the surface response is dominated by the stiffness and that the damping term is of secondary importance. Hence in (4.11) we take

$$R = 0 \quad \text{and} \quad X = (1/\omega)\,(K - M\omega^2), \tag{4.12}$$

where for spring-dominated effects it is assumed that

$$K - M\omega^2 > 0. \tag{4.13}$$

In this limit the response of the plate is governed by the purely local relation

pressure difference = impedance × velocity.

The structure lies in equilibrium along ($x_3 = 0$, $x_1 < 0$), and as in the previous models we are concerned with the determination of the motion – one-dimensional on the surface, two-dimensional in the fluid – generated when a subsonic surface wave is normally incident from $x_1 = -\infty$ on the edge at $x_1 = 0$. The total surface deflection in the x_3-direction is expressed as a sum of incident and scattered parts, that is, $\eta_\mathrm{i} + \eta$, where the incident is the bounded solution of the pertinent doubly-infinite coupled problem and η is the scattered wave. For the particular lumped parameter representation envisaged here

$$\eta_i = d \exp(\mathrm{i}\alpha x_1), \tag{4.14}$$

with the free wavenumber $\qquad \alpha = (k_0^2 + \bar{\mu}^2)^{\frac{1}{2}}, \quad \alpha > 0, \qquad (4.15)$

where $\qquad \bar{\mu} = 2\rho\omega^2/(K - M\omega^2) \sim 2\rho\omega^2/K \quad \text{by (4.13)}. \qquad (4.16)$

Likewise we write the total potential, an odd function of x_3, as $\phi_\mathrm{i} + \phi$, where

$$\phi_\mathrm{i} = (\mathrm{i}\omega d/\gamma_\alpha) \exp(\mathrm{i}\alpha x_1 - \gamma_\alpha |x_3|) \operatorname{sgn} x_3, \tag{4.17}$$

with $\gamma_\alpha = (\alpha^2 - k_0^2)^{\frac{1}{2}} > 0$, ϕ_i the potential induced by the incident wave η_i, and ϕ the scattered field in the fluid. This problem requires the determination of a scattered potential and deflection, each satisfying appropriate conditions at infinity and on the plate. Introduction of half-range Fourier transforms leads to an unexceptional Wiener–Hopf equation, with kernel

$$M(k) = (k^2 - k_0^2)^{\frac{1}{2}} - \bar{\mu}; \tag{4.18}$$

expressions for the reflected structural wave and distant radiated field are readily obtained from the Wiener–Hopf equation.

Specifically, it emerges that as $x_1 \to -\infty$ on the structure the sole propagating reflected wave assumes the form

$$\eta_\mathrm{r}(x_1) = Rd \exp(-\mathrm{i}\alpha x_1), \tag{4.19}$$

in which the reflection coefficient, R, is given by

$$R = (\bar{\mu}/2\alpha^2)\, M_+^2(\alpha). \tag{4.20}$$

In problems of this type, the factorization (in a Wiener–Hopf sense) of the dispersion function is paramount. Even in the simplest coupled configurations this is a matter of some technical

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difficulty. The calculation of the edge-scattering efficiency, for example, requires explicit expressions for $|M_+(\alpha)|$ and $|M_-(-k_0 \cos\theta)|$, and the two wavenumbers involved are often disparate. However, an exact expression for $|R|$ valid for arbitrary values of k_0 and $\bar{\mu}$ may be obtained in the current problem from the Cauchy integrals that define the Wiener–Hopf factors; this is, of course, unusual, and not paralleled by the expression for the phase shift, which is extremely complicated. Details are relegated to Appendix 3 to which the reader is referred. Here we quote only the two crucial results, namely

$$|M_+(\alpha)| = (2\alpha)^{\frac{1}{2}}, \tag{4.21}$$

$$|M_-(-k_0 \cos\theta)| = (\alpha + k_0 \cos\theta)^{\frac{1}{2}}. \tag{4.22}$$

Substitution of the first result in (4.20) leads to

$$|R| = (1 + k_0^2/\bar{\mu}^2)^{-\frac{1}{2}}, \tag{4.23}$$

from which it follows that $$|R| = 1, \quad k_0 = 0,$$

while $$|R| < 1, \quad k_0 > 0.$$

By analogy with (4.6) we see that the energy of the subsonic pressure wave is diminished by an amount

$$\rho\omega^3 d^2 k_0^2 / 2\bar{\mu}^3 (\bar{\mu}^2 + k_0^2)^{\frac{1}{2}} \tag{4.24}$$

on reflection.

The distant radiated acoustic field, associated with the steepest descents integral, is given by the cylindrical wave

$$\phi_{\mathrm{d}} \sim r_*^{-\frac{1}{2}} \exp(\mathrm{i}k_0 r_*)\, \Phi(\theta), \tag{4.25}$$

with angular dependence

$$\Phi(\theta) = \left(\frac{k_0}{2\pi}\right)^{\frac{1}{2}} \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi} \sin\theta\, \omega d\, M_+(\alpha)\, \mathrm{sgn}\, x_3}{\gamma_\alpha(\alpha - k_0\cos\theta)\, M_-(-k_0\cos\theta)}. \tag{4.26}$$

Substitution of the relevant expressions for $M_+(\alpha)$ and $M_-(-k_0\cos\theta)$ in (4.21), (4.22) allows the power radiated to infinity by the cylindrical wave to be evaluated from (4.9) in terms of elementary functions. Indeed, for double-sided fluid loading the power radiated to infinity by the bulk wave is

$$\rho\omega^3 d^2 k_0^2 / 2\bar{\mu}^3 (\bar{\mu}^2 + k_0^2)^{\frac{1}{2}}.$$

This vanishes when $k_0 = 0$ and precisely equals the energy lost in the reflection process for arbitrary values of k_0 and $\bar{\mu}$.

The scattering efficiency of the edge – the ratio of scattered acoustic power to incident wavepower – follows readily and is given by

$$e_{\mathrm{s}} = k_0^2/\bar{\mu}^2\, (1 + k_0^2/\bar{\mu}^2)^{-1}, \tag{4.27}$$

which is small when $k_0 \to 0$. However, the small, non-zero value of k_0 is essential in providing the sole mechanism for transporting a small amount of energy to the body of the fluid; this amount of energy exactly balances the structural energy lost in the reflection process when the surface is passive and no other loss mechanisms are allowed for. We have not been able to prove this self-evident truth in the case of the thin elastic plate possessing flexural rigidity and inertia, but we believe that the simpler model with a constant stiffness term provides a totally convincing demonstration of the energy balance in the reflection process.

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5. An approximate method for very heavy fluid loading: the finite circular plate with eccentric drive

5.1. *General asymptotic method for very heavy fluid loading*

The first entry of table 1 for the semi-infinite reflection of a structural wave was used in Crighton & Innes (1983) to determine the response of a finite unbaffled strip plate with free edges driven by a line force $F_0 \exp(-\mathrm{i}\omega t)$. Analogous methods may be used to analyse other simple finite geometries subject to alternative types of forcing, for example, localized point drive. This can be done easily when the lengths, l, of the plates in question are large compared with the fluid loading wavelength κ^{-1} (that condition leading to $k_\mathrm{p}\, l N^{-\frac{1}{5}} \gg 1$; a limit readily satisfied in the very heavy fluid-loading limit assumed herein). It is then a straightforward matter to show that the simplicity of the results of Crighton & Innes (1983) for mode shapes and natural frequencies is *general* and not a particular outcome of the specific problem discussed.

Quite generally the expression for the plate response must contain a term (the Green function) corresponding to the infinite plate response, and this term can be suitably chosen to incorporate completely the singularity associated with a given forcing. The response must also include terms to represent the free waves demanded by the presence of the edges. At distances from any given edge that are small compared with l but still large compared with κ^{-1}, the response must be expressible as a sum of terms representing waves incident on, and reflected from, that edge. Now the reflection process is a purely local one and so it is permissible to relate the incident and reflected waves by using the semi-infinite configuration results of table 1. Any particular combination of edge and boundary conditions may be accommodated simply by choosing the value of Θ (the phase change on reflection) accordingly. Thus the strip plate dealt with by Crighton & Innes (1983) may be generalized to arbitrary edge and boundary conditions, as we now show.

Specifically we reconsider the plate lying in $x_3 = 0$, $|x_1| < l$, $-\infty < x_2 < \infty$, driven by the line force $F_0 \exp(-\mathrm{i}\omega t)$ along the x_2-axis. The plate velocity, an even function of x_1, assumes the form (argued to hold everywhere on the plate except within a distance $O(\kappa^{-1})$ of each edge)

$$V(x_1) = -\frac{\mathrm{i}\omega F_0}{2\pi B}\int_{-\infty}^{\infty} \frac{\gamma \exp(\mathrm{i}k|x_1|)\,\mathrm{d}k}{(k^4-k_\mathrm{p}^4)\,\gamma - \mu k_\mathrm{p}^4} + A\cos(\kappa x_1) + B\cosh(\kappa x_1), \tag{5.1}$$

in which the individual terms assume the roles already cast. For a discussion of the first (Green function) term, see for example, Crighton (1983), Crighton & Innes (1983). By the straightforward arguments given there, the behaviour near the edge $x = l$ emerges as

$$V(x_1) \sim (\omega F_0/5B\kappa^3 + \tfrac{1}{2}A)\,\mathrm{e}^{\mathrm{i}\kappa l}\,\mathrm{e}^{-\mathrm{i}\kappa(l-|x_1|)} + \tfrac{1}{2}A\,\mathrm{e}^{-\mathrm{i}\kappa l}\,\mathrm{e}^{\mathrm{i}\kappa(l-|x_1|)} + \tfrac{1}{2}B(\mathrm{e}^{\kappa l}\,\mathrm{e}^{-\kappa(l-|x_1|)} + \mathrm{e}^{-\kappa l}\,\mathrm{e}^{\kappa(l-|x_1|)}), \quad \kappa l \gg 1,$$

from which follows

$$\left.\begin{aligned} B &= 0 \quad (\text{since } \kappa l \gg 1),\\ \text{and} \quad A &= A_0^l F_0 \\ &= (\omega F_0/5B\kappa^3)\,[-1 + \mathrm{i}\cot(\tfrac{1}{2}\Theta + \kappa l)], \end{aligned}\right\} \tag{5.2}$$

on application of the reflection rule represented by $R = \exp(\mathrm{i}\Theta)$ and Θ is the phase change relevant to the particular configuration under consideration. This determines the field everywhere except within a region of extent κ^{-1} around each edge where the 'acoustic' (or

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more strictly, 'hydrodynamic') terms must be retained. In this region the structural field is determined by the full solution to the semi-infinite Wiener–Hopf problem as given in §2 and it comprises incident and reflected plane waves together with a 'hydrodynamic' field (associated with the near incompressibility of the flow) of amplitude comparable with those of the plane waves. We note that, in obtaining such simple results as follow, the heavy fluid-loading effects were retained (i) in the behaviour near the drive point, where the integral contributes the infinite plate line drive admittance, (ii) in the free wavenumber κ of the free waves described by the complementary functions in (5.1), and (iii) in the determination of $\mathscr{A}_0^l$ from the semi-infinite analysis.

As to the status of expressions derived in this way, we show in a later paper (Crighton & Innes 1984) that (5.1) with A and B given by (5.2), is not merely a convenient approximation of some engineering utility, but that it is in fact an asymptotic solution to the full problem in a precisely defined sense. One is thus encouraged to apply the method to large, but finite, geometries where there has been, up to the present, no possibility of any rational approach.

Returning to the general strip plate problem, the line drive admittance may be written

$$\mathscr{A}_0 = \mathscr{A}_0^{\infty} + \mathscr{A}_0^l \tag{5.3}$$

where $\mathscr{A}_0^{\infty}$ is the now familiar infinite plate result (Crighton 1972). We then find that

$$\mathscr{A}_0 = (\mathrm{i}\omega/5B\kappa^3)\,[\tan\tfrac{1}{10}\pi + \cot(\tfrac{1}{2}\Theta + \kappa l)]. \tag{5.4}$$

This expression displays the same features as were noted by Crighton & Innes (1983) in one particular case: notably that $\mathrm{Re}\,\mathscr{A}_0 = 0$, which implies negligible acoustic radiation loss in the edge reflection process (and no other loss mechanisms are allowed for); and that the reactive component of the admittance can assume all possible signs and magnitudes corresponding to appropriate values of $\kappa l + \frac{1}{2}\Theta$. In particular it follows that when

$$\kappa l = (n - \Theta/2\pi)\,\pi \quad (\text{integer } n), \tag{5.5}$$

we can let $F_0 \to 0$, $\mathscr{A}_0^l \to \infty$ so that $\mathscr{A}_0^l F_0$ remains an arbitrary finite constant. This leads to *free modes of oscillation* of the finite strip plate under heavy fluid loading with *mode shapes* (away from the edge) given by

$$\cos\kappa x_1 = \cos[(n - \Theta/2\pi)\,\pi x_1/l]. \tag{5.6}$$

The eigenvalue equation (5.5) gives rise to natural frequencies of the form

$$\omega_n = (n - \Theta/2\pi)^2\,\pi^2 N^{\frac{2}{5}}\,(B/ml^4)^{\frac{1}{2}}, \tag{5.7}$$

the last two expressions being very close in form to the corresponding ones for vacuum dynamics.

The line force excitation originally introduced here provokes only a symmetric response (in x_1), with cosine free modes. A central moment excitation would provoke a purely antisymmetric response, with sine mode shapes whose characteristics can readily be found as above. In general the response will be neither even nor odd, and general initial conditions will cause both odd and even modes to be excited.

As to the sound generated in the far field by the forced motion of the finite plate, much here depends on the edge and baffle conditions. For the free unbaffled plate it was shown by Crighton & Innes (1983) that the acoustic field generated by scattering at the edges is a purely numerical

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constant (independent of ϵ, but dependent, of course, on θ) times $N^{\frac{4}{5}}$ times the field radiated by the line force acting on an infinite plate, and therefore the latter field dominates; this makes the acoustic calculations very simple. That remains true for the other two unbaffled edge cases for which details are set out in table 1. For baffled edges the corresponding factor appears to be $N^{-\frac{3}{5}}$, indicating that in such cases it is the edge-scattering mechanism that provides the dominant far-field signal rather than the primary excitation itself. It is straightforward to calculate the far field scattered by the two edges from the solution of the semi-infinite problems; all one has to do is multiply the field for one edge by $\cos(k_0 l \cos\theta)$ which is not necessarily close to unity in the limit envisaged.

5.2. *Circular plate with eccentric drive*

Of considerable interest is the extension to circular and rectangular plates of the calculations for mode shapes and resonance frequencies, and, more generally, of the response of such plates to arbitrary forcing. For circular plates the extension is straightforward. Suppose the plate occupies $r \leqslant a$, with point drive at $(r_0, 0)$ in cylindrical polar coordinates (r, ϕ). Then the infinite plate response is

$$v_\infty = -\frac{\mathrm{i}\omega F_0}{4\pi B}\int_{-\infty}^{+\infty}\frac{\gamma H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}_0|)\,k\,\mathrm{d}k}{(k^4-k_\mathrm{p}^4)\,\gamma-\mu k_\mathrm{p}^4} \tag{5.8}$$

(replacing the Green function of (5.1) for line excitation), and for $\kappa|\boldsymbol{r}-\boldsymbol{r}_0| \gg 1$ the dominant term comes from the plot at $k=\kappa$ where, as usual, $\kappa = k_\mathrm{p} N^{-\frac{1}{5}}$ is the free wavenumber in the heavy fluid-loading limit. Thus

$$v_\infty \sim \frac{\omega F_0}{10B\kappa^2} H_0^{(1)}(\kappa|\boldsymbol{r}-\boldsymbol{r}_0|) \tag{5.9}$$

$$= \frac{\omega F_0}{10B\kappa^2}\sum_{m=0}^{\infty}\epsilon_m \cos m\phi \begin{cases} J_m(\kappa r)\, H_m^{(1)}(\kappa r_0), & r<r_0, \\ H_m^{(1)}(\kappa r)\, J_m(\kappa r_0), & r>r_0, \end{cases}$$

by a well known expansion theorem, ϵ_m being the Neumann symbol ($\epsilon_0 = 1$, $\epsilon_m = 2$, $m \geqslant 1$). Assuming that the drive point is not located very close to the edge, we can write the infinite plate velocity near the edge as the sum over azimuthal mode number m of $(\omega F_0 \epsilon_m \cos m\phi)/10B\kappa^2$ times $(2/\pi\kappa a)^{\frac{1}{2}} J_m(\kappa r_0) \exp[-\frac{1}{4}\pi\mathrm{i} - \frac{1}{2}m\pi\mathrm{i} + \mathrm{i}\kappa a] \exp[\mathrm{i}\kappa(r-a)]$.

Now the effects of the edges must be expressible as sums of regular plane waves of wavenumber κ, or equivalently, in the present geometry, sums of terms like $\cos m\phi\,\{J_m(\kappa r)$, $Y_m(\kappa r), I_m(\kappa r), K_m(\kappa r)\}$, with standard Bessel function notation. Of these only the first is allowed, for the Y_m, K_m terms have singularities at $r=0$ (and the Green function *completely* accounts for those singularities) while the I_m terms become large as $r \to \infty$ (like the hyperbolic cosine terms excluded from the strip plate for the same reason). Therefore add to (5.9)

$$v_\mathrm{c} = \sum_{m=0}^{\infty} A_m \epsilon_m \cos m\phi\, J_m(\kappa r), \tag{5.10}$$

to represent the effect of the edges, and expand up the J_m function near $r=a$, so that there the total field (with sum over $\epsilon_m \cos m\phi$ suppressed) is

$$(\omega F_0/10B\kappa^2)\,(2/\pi\kappa a)^{\frac{1}{2}} J_m(\kappa r_0)\exp[-\tfrac{1}{4}\pi\mathrm{i}-\tfrac{1}{2}m\pi\mathrm{i}+\mathrm{i}\kappa a]\exp[\mathrm{i}\kappa(r-a)]$$
$$+A_m\tfrac{1}{2}(2/\pi\kappa a)^{\frac{1}{2}}\{\exp[\mathrm{i}\kappa(r-a)+\mathrm{i}\kappa a-\tfrac{1}{2}m\pi\mathrm{i}-\tfrac{1}{4}\pi\mathrm{i}]+\exp[-\mathrm{i}\kappa(r-a)-\mathrm{i}\kappa a+\tfrac{1}{2}m\pi\mathrm{i}+\tfrac{1}{4}\pi\mathrm{i}]\}. \tag{5.11}$$

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Then we argue that each azimuthal mode is reflected from the edge of a large plate as in the corresponding semi-infinite problem for purely normal incidence, so that the coefficient of $\exp[-i\kappa(r-a)]$ in (5.11) must be just $\exp(i\Theta)$ times that of $\exp[i\kappa(r-a)]$. If $F_0 \neq 0$ the resulting equation determines A_m in terms of F_0, while if $F_0 = 0$ the resulting equation is the eigenvalue equation determining the natural frequencies of an mth azimuthal mode, which then has mode shape $\cos m\phi\, J_m(\kappa r)$.

The expression for A_m is

$$A_m = (\omega F_0/10B\kappa^2)\, J_m(\kappa r_0)\,[-1+i\cot(\lambda_m+\tfrac{1}{2}\Theta)], \tag{5.12}$$

where

$$\lambda_m = \kappa a-(m+\tfrac{1}{2})\,\tfrac{1}{2}\pi, \tag{5.13}$$

and gives the result

$$\mathscr{A} = \mathscr{A}_\infty + \frac{\omega}{10B\kappa^2}\sum_{m=0}^{\infty}\epsilon_m J_m^2(\kappa r_0)\,[-1+i\cot(\lambda_m+\tfrac{1}{2}\Theta)],$$

for the drive admittance, $\mathscr{A}_\infty$ being the infinite plate contribution. But for any z

$$\sum_{m=0}^{\infty}\epsilon_m J_m^2(z) = 1,$$

and hence

$$\mathscr{A} = \left[\frac{1}{8(Bm)^{\frac{1}{2}}}\right]\tfrac{4}{5}iN^{\frac{2}{5}}\left[-\tan\tfrac{1}{10}\pi+\sum_{m=0}^{\infty}\epsilon_m J_m^2(\kappa r_0)\cot(\lambda_m+\tfrac{1}{2}\Theta)\right]. \tag{5.14}$$

It is again satisfying that $\operatorname{Re}\mathscr{A} = 0$, and we have a simple result for the drive admittance which obviously displays all the features previously referred to and contains, in addition, dependence on excitation position r_0 and mode index m. In the simplest case, $r_0 = 0$, the whole vibration pattern is axisymmetric and (5.14) reduces to

$$\mathscr{A} = \left[\frac{1}{8(Bm)^{\frac{1}{2}}}\right]\tfrac{4}{5}iN^{\frac{2}{5}}\,[-\tan\tfrac{1}{10}\pi+\cot(\kappa a-\tfrac{1}{4}\pi+\tfrac{1}{2}\Theta)], \tag{5.15}$$

which is similar to (5.4) and exhibits the same features. As these features have not been itemized in the present paper, we note them here; if,

(i) $[-\tan\tfrac{1}{10}\pi+\cot(\kappa a-\tfrac{1}{4}\pi+\tfrac{1}{2}\Theta)] > 0$, $\operatorname{Im}\mathscr{A} > 0$, corresponding to *mass loading*, whereas if

(ii) $[-\tan\tfrac{1}{10}\pi+\cot(\kappa a-\tfrac{1}{4}\pi+\tfrac{1}{2}\Theta)] < 0$, $\operatorname{Im}\mathscr{A} < 0$, corresponding to *stiffness loading*, while if

(iii) $[-\tan\tfrac{1}{10}\pi+\cot(\kappa a-\tfrac{1}{4}\pi+\tfrac{1}{2}\Theta)] = 0$, $\operatorname{Im}\mathscr{A} = 0$ and there is *anti-resonance* (zero velocity) at the drive point, whereas when

(iv) $\sin(\kappa a-\tfrac{1}{4}\pi+\tfrac{1}{2}\Theta) = 0$, $\operatorname{Im}\mathscr{A} = \infty$ and there is *resonant response*, while finally if

(v) $\cos(\kappa a-\tfrac{1}{4}\pi+\tfrac{1}{2}\Theta) = 0$, there is what may be called '*transparency*' in that the drive point behaviour, as measured by $\operatorname{Im}\mathscr{A}$, is unaffected by the presence of the edges of the plate.

It is easy to see from (5.12) that the eigenvalue equation for free oscillations with no forcing is

$$\lambda_m+\tfrac{1}{2}\Theta = n\pi$$

and then from (5.10) that the corresponding mode shapes are

$$\cos m\phi\, J_m(\kappa r).$$

This is as far as it is worthwhile to take discussion of the circular plate. Another case of considerable practical interest involves the rectangular plate. Here, although in principle the

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same ideas work, the situation is considerably more complicated. On each edge is incident *not* a single plane wave at normal incidence, but a continuum of plane waves at all angles permitted by the geometry of the plate and the point of excitation. What is required then is a knowledge of the reflection coefficient for semi-infinite geometry as a function of the angle of incidence of an obliquely incident wave. Both this problem and its application to finite rectangular plates, are of some complexity and it is unlikely at present that, even in the case of the vanishing of the Poisson ratio for the material of the plate, *simple* analytical expressions for the reflection coefficient will be forthcoming. However, a satisfactory treatment of the analogous problem involving the reflection of waves obliquely incident on the edge of a semi-infinite membrane is possible; this is the subject of the next section, which opens the way to the solution of many problems, of great practical importance, involving finite panels with fixed or free edges and subject to localized forcing.

6. The reflection of an obliquely incident plane wave at the edge of a semi-infinite wavebearing surface

This section contains the generalization of the unbaffled semi-infinite configuration of §2 to the case in which the incoming structural wave is not normally incident on the edge at $x_1 = 0$, but strikes it at an oblique angle, $\frac{1}{2}\pi - \theta_0$, say. We discuss only the *reflection* of a single subsonic surface wave at the edge of an elastic membrane. We believe that this is a reasonable modelling procedure which can be expected to predict, albeit qualitatively, the dominant physical mechanism in similar problems concerned with surfaces whose motion is governed by equations of greater mathematical complexity.

A formal analysis is presented for the configuration in which the membrane is unbaffled; specific results of simple, though not simplistic, form are then readily obtained in the low frequency–heavy fluid loading limit and when the edge is either fixed or free.

6.1. *General analysis for reflection of structural waves at oblique incidence on an edge*

We consider, then, a homogeneous elastic membrane of surface mass m per unit area under uniform tension T. The membrane occupies the region $x_1 < 0$ of the (x_1, x_2) plane and is totally immersed in static, inviscid fluid. Appropriate incident fields, chosen to represent a subsonic surface wave travelling over the membrane towards the edge $x_1 = 0$ and the attendant fluid potential, are given by

$$\eta_\mathrm{i} = d \exp\left[\mathrm{i}\kappa \cos\theta_0 x_1 + \mathrm{i}\kappa \sin\theta_0 x_2 - \mathrm{i}\omega t\right], \tag{6.1}$$

$$\phi_\mathrm{i} = (\mathrm{i}\omega d/\gamma_\kappa) \exp\left[\mathrm{i}\kappa \cos\theta_0 x_1 + \mathrm{i}\kappa \sin\theta_0 x_2 - \mathrm{i}\omega t\right] \exp\left[-\gamma_\kappa |x_3|\right] \operatorname{sgn} x_3, \tag{6.2}$$

wherein κ is the free wavenumber at radian frequency ω in the doubly-infinite coupled system. The angle of incidence, θ_0, is measured between the normal to the edge in the surface and the direction of travel at incidence and thus $\theta_0 = 0$ corresponds to normal incidence at the edge. For a single fixed x_2-wavenumber component, $\kappa \sin\theta_0$, we assume that the *total* membrane deflection may be written as $\eta_\mathrm{i} + \eta(x_1) \exp\left[\mathrm{i}\kappa \sin\theta_0 x_2 - \mathrm{i}\omega t\right]$ and we associate with this a *total* fluid potential $\phi_\mathrm{i} + \phi(x_1, x_3) \exp\left[\mathrm{i}\kappa \sin\theta_0 x_2 - \mathrm{i}\omega t\right]$. The transverse vibratory field on the membrane is governed by the equation

$$T(\nabla_\mathrm{s}^2 + k_\mathrm{m}^2)\, \eta(x_1) \exp\left[\mathrm{i}\kappa \sin\theta_0 x_2\right] = 2p(x_1, 0_+), \quad x_1 < 0, \quad x_3 = 0, \tag{6.3}$$

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where ∇_s^2 is the surface Laplacian and where the fluid overpressure p is related to the scattered potential via $p = \rho i\omega\phi$. In addition, the interaction at the fluid interface is expressible as

$$-i\omega\eta(x_1) = \partial\phi(x_1)/\partial x_3, \quad x_1 < 0, \quad x_3 = 0, \tag{6.4}$$

while the symmetry condition for the unbaffled configuration is equivalent to

$$\phi(x_1) = -(i\omega d/\gamma_\kappa)\exp[i\kappa x_1\cos\theta_0], \quad x_1 > 0, \quad x_3 = 0. \tag{6.5}$$

Finally, the scattered potential in the fluid medium is required to satisfy the Helmholtz equation

$$(\nabla^2 + k_0^2)\,\phi(x_1, x_3)\exp[i\kappa\sin\theta_0 x_2] = 0, \tag{6.6}$$

together with a radiation or extinction condition as $|x_3| \to \infty$ in the fluid.

The problem posed above is solved by straightforward application of Jones's technique (Noble 1958, p. 52ff); yet again we refrain from any discussion of the niceties of the procedure, but simply state that the equations lead to a standard form of Wiener–Hopf functional equation,

$$\begin{aligned} K(k, \sin\theta_0)\,[\tilde{\phi}_-(k,0) + \omega d/\gamma_\kappa(k+\cos\theta_0)] \\ +\{[k^2-(k_m^2-\kappa^2\sin^2\theta_0)]\,(\partial\tilde{\phi}_+/\partial x_3)(k,0) + i\omega P(k)\} \\ +(2\rho\omega^2/T)\,[\omega d/\gamma_\kappa(k+\kappa\cos\theta_0)] = 0, \quad k\in R_+\cap R_-, \end{aligned} \tag{6.7}$$

with $P(k) = [ik\eta(0) - \partial\eta(0)/\partial x_1]$. Which of the two constants included in $P(k)$ is known, depends on the prescribed edge constraints: for example, if the edge is fixed at $x_1 = 0$ it follows that the total displacement there is zero, which implies that

$$\eta(0) = -d, \tag{6.8}$$

with $\partial\eta(0)/\partial x_1$ to be determined; whereas for a free edge it is required that the net force there is zero, which implies that

$$\partial\eta(0)/\partial x_1 = -id\kappa\cos\theta_0, \tag{6.9}$$

with $\eta(0)$ an unknown constant to be found in the course of the work. The kernel,

$$K(k, \sin\theta_0) = [k^2-(k_m^2-\kappa^2\sin^2\theta_0)][k^2-(k_0^2-\kappa^2\sin\theta_0)]^{\frac{1}{2}} - 2\rho\omega^2/T, \tag{6.10}$$

is the generalization to oblique incidence of the familiar membrane–fluid dispersion function, $[(k^2-k_m^2)\,(k^2-k_0^2)^{\frac{1}{2}} - 2\rho\omega^2/T]$. Furthermore, we recall that the usual conditions limiting the growth of ϕ obtain as $|x_3| \to \infty$ in the fluid, and the branch cuts associated with the square root function, $[k^2-(k_0^2-\kappa^2\sin^2\theta_0)]^{\frac{1}{2}}$ are located in accordance with this requirement.

Now we proceed formally by writing

$$K(k, \sin\theta_0) = K_+(k)\,K_-(k),$$

where the factors $K_\pm$ are analytic and non-zero in $R_\pm$ respectively, with $K_\pm(k) = O(|k|^{\frac{3}{2}})$ as $|k| \to \infty$ in the appropriate half-planes and where, additionally, (2.11) is satisfied ($K_+(-k) = K_-(k)$). The required splitting is then trivial and results in a pair of algebraic equations, namely

$$\tilde{\phi}_-(k,0) = \frac{E(k)}{K_-(k)} - \frac{\omega d}{\gamma_\kappa(k+\kappa\cos\theta_0)}\left[1 - \frac{K_+(\kappa\cos\theta_0)}{K_-(k)}\right], \tag{6.11}$$

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and

$$\partial\tilde{\phi}_+(k,0)/\partial x_3 = \{-E(k)\,K_+(k) - \mathrm{i}\omega P(k) - [\omega d/\gamma_\kappa(k+\kappa\cos\theta_0)]\,[K_+(\kappa\cos\theta_0)\,K_+(k) + 2\mu k_{\mathrm{m}}^2]\}/[k^2-(k_{\mathrm{m}}^2-\kappa^2\sin^2\theta_0)]. \quad (6.12)$$

The split equations contain an unknown entire function, $E(k)$, which is introduced via the invocation of the extended form of Liouville's theorem. In the present problem we assume that the standard conditions hold at the edge, namely that the fluid pressure is bounded and that the velocity has, at worst, an inverse square root singularity. Then it is evident that $E(k)$ is a constant, E_0, say, and subsequent removal of apparent poles in the domains of analyticity of $\partial\tilde{\phi}_+(k,0)/\partial x_3$ and $\tilde{\eta}_-(k)$ leads to a pair of equations which suffice to determine uniquely both unknown constants. In this unbaffled geometry the two equations obtained thereby can both be written in the form

$$E_0 K_+(k) + [\omega d/\gamma_\kappa(k+\kappa\cos\theta_0)]\,[2\mu k_{\mathrm{m}}^2 + K_+(\kappa\cos\theta_0)\,K_+(k)] = -\mathrm{i}\omega\,P(k), \quad (6.13)$$

in which $k = \pm(k_{\mathrm{m}}^2-\kappa^2\sin^2\theta_0)^{\frac{1}{2}}$ in turn, and $K_+(-k)$ is understood to mean $K(k,\sin\theta_0)/K_-(-k)$. This, then, completes the formal determination of the field. However, we are concerned here with one particular aspect of the scattered field, namely that of the form of the reflected wave on the membrane. The structural field can be written down simply as an inverse Fourier transform of known integrand

$$\tilde{\eta}_-(k) = \left\{-P(k) - \frac{2\rho\mathrm{i}\omega}{T}\left[E_0 + \frac{\omega d\,K_+(\kappa\cos\theta_0)}{\gamma_\kappa(k+\kappa\cos\theta_0)}\right]\frac{1}{K_-(k)} + \frac{2\rho\mathrm{i}\omega}{T}\,\frac{\omega d}{\gamma_\kappa(k+\kappa\cos\theta_0)}\right\}\Big/[k^2-(k_{\mathrm{m}}^2-\kappa^2\sin^2\theta_0)]. \quad (6.14)$$

Because of (6.13), $\tilde{\eta}_-(k)$ has no pole at $k = -(k_{\mathrm{m}}^2-\kappa^2\sin\theta_0)^{\frac{1}{2}}$; nor is the pole at $k = +(k_{\mathrm{m}}^2-\kappa^2\sin^2\theta_o)^{\frac{1}{2}}$ genuine. Therefore we deform the contour of integration into the upper half-plane where the contribution from the semi-circle at infinity is zero. The procedure is standard and we simply quote the result that the only singularity which gives a reflected propagating wave in the surface as $x_1 \to -\infty$ is that located at $k = \kappa\cos\theta_0$, and the associated residue contribution leads to a reflected field on the membrane,

$$\eta_{\mathrm{r}}(x_1) = \frac{2\mu k_{\mathrm{m}}^2}{\omega}\left[E_0 + \frac{\omega d\,K_+(\kappa\cos\theta_0)}{\gamma_\kappa 2\kappa\cos\theta_0}\right]\frac{K_+(\kappa\cos\theta_0)\exp[-\mathrm{i}\kappa\cos\theta_0 x_1]}{(\kappa^2-k_{\mathrm{m}}^2)\,[\mathrm{d}K(k,\sin\theta_0)/\mathrm{d}k]_{k=\kappa\cos\theta_0}}. \quad (6.15)$$

It is not worthwhile to pursue the general analysis further; instead we return to the heavy fluid-loading limit and consider the solution of the posed problem subject to the two specific edge constraints mentioned earlier.

6.2. *Specific results and discussion*

First, we suppose that the edge at $x_1 = 0$ is free: if we assume that the condition for heavy fluid loading is achieved, then $M = \epsilon N$ with $N \ll 1$ and the free wavenumber κ attains the value $k_{\mathrm{m}}/N^{\frac{1}{3}}$. Therefore it follows from (6.9) that

$$\partial\eta(0)/\partial x_1 \sim -\mathrm{i}dk_{\mathrm{m}}\cos\theta_0/N^{\frac{1}{3}}. \quad (6.16)$$

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In this limit the equations (6.13) for E_0 and $\eta(0)$ have a simple solution that can be incorporated into (6.15) to give

$$\eta_r(x_1) \sim -\frac{dN}{k_m^3}\frac{K_+^2(k_m\cos\theta_0/N^{\frac{1}{3}})}{6\cos^2\theta_0} \times\left[\frac{k_m^3\,e^{2i\theta_0}-N\,e^{-2i\theta_0}\,K_+^2(ik_m\sin\theta_0/N^{\frac{1}{3}})}{k_m^3-NK_+^2(ik_m\sin\theta_0/N^{\frac{1}{3}})}\right]\exp\left[-i\kappa\cos\theta_0 x_1\right]. \quad (6.17)$$

In the heavy fluid-loading limit it is anticipated that the incident wave will be reflected without change in magnitude, to leading order at any rate; straightforward substitution of the results of the Appendix and a little manipulation reveal that this is indeed the case. Specifically, for a free edge in the low frequency–heavy fluid loading limit, it emerges that

$$\eta_r(x_1) \sim d\exp\left(-i\kappa\cos\theta_0 x_1\right)\exp\left(i\Theta_1\right), \quad (6.18)$$

where it is convenient to define the real angle Θ_1 via the auxiliary function

$$\Theta_\mu = -2\cos\theta_0 I(1)+\arctan\left\{\frac{[1-\exp\left(4\sin\theta_0 I(0)\right)]\sin 2\theta_0}{[1+\exp\left(4\sin\theta_0 I(0)\right)]\cos 2\theta_0+2\mu\exp\left(2\sin\theta_0 I(0)\right)}\right\}, \quad (6.19)$$

with $\mu = 1$ and $I(\lambda)$ as defined in the Appendix (A 78). We recall that for normal incidence $\theta_0 = 0$, and then the angle of phase shift is given by

$$\Theta_1 = -\frac{1}{\pi}\int_0^\infty \frac{\arctan u^{\frac{3}{2}}\,du}{u^{\frac{1}{2}}(u+1)}. \quad (6.20)$$

Evaluation of this integral shows that the corresponding reflection coefficient R is $e^{-\frac{1}{4}i\pi}$, in agreement with the value obtained by independent means in the fashion of §2, for the heavy fluid-loading problem that involves a membrane with a free edge.

The second case, in which the edge is fixed, does not differ radically from the previous one; details are given by Innes (1983). We find that, for heavy fluid loading

$$\eta_r(x_1) \sim d\exp\left[-i\kappa\cos\theta_0 x_1\right]\exp\left(i\Theta_{-1}\right), \quad \theta_0 \neq 0, \quad (6.21)$$

wherein the angle, Θ_{-1}, is the auxiliary function of (6.19), but now with $\mu = -1$. For normal incidence, $\theta_0 = 0$, we find that the value of the reflection coefficient is indeterminate. However, a straightforward limiting process shows that the reflection coefficient is given then by

$$R = e^{-\frac{11}{12}\pi i}$$

for a wave normally incident on the fixed edge of an unbaffled membrane subject to heavy fluid loading. This value has been verified by independent means.

The specific results of this section have been confined to the range of fluid loading defined by $M = \epsilon N$, $N \ll 1$: the so called 'very heavy fluid-loading' limit. In the two cases considered here in some detail and which involve an unbaffled membrane with either a free or fixed edge, we have shown that in the stated limit the general result of §2 is still valid: an incident surface wave $\eta_i = d\exp\left[i\kappa\cos\theta_0 x_1+i\kappa\sin\theta_0 x_2\right]$ is reflected from the edge of a semi-infinite membrane generating a cylindrical 'acoustic' field in the process. The resulting field on the membrane comprises coupled modes arising from the pole contributions associated with the zeros of $K(k,\sin\theta_0)$ in R_+ and an algebraically decaying branch line contribution to the inverse Fourier integral which determines the structural response. At great distances from the edge the sole

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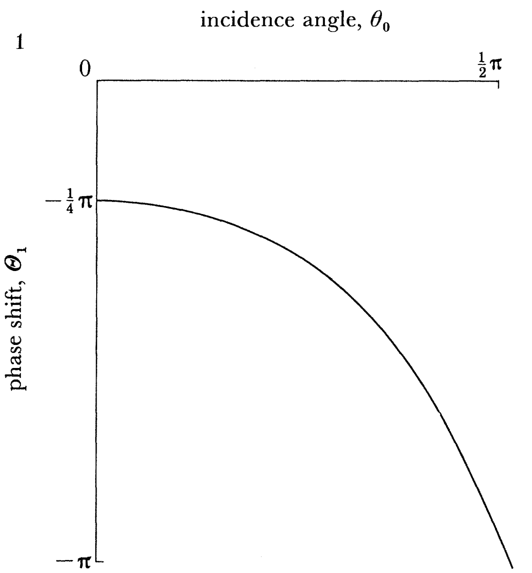
propagating wave is that arising from the single real positive zero of (6.10), at $k = \kappa \cos\theta_0$, and this wave assumes the form

$$\eta_r = Rd \exp\left[-i\kappa \cos\theta_0 x_1 + i\kappa \sin\theta_0 x_2\right].$$

For the two postulated edge constraints it has been shown that, to leading order at least, the reflection coefficient R merely assumes the form of a phase shift, $e^{i\Theta_\mu}$, with $\mu = \pm 1$ corresponding to the unbaffled configuration with free and fixed edges respectively. Moreover, it is still possible to write down an expression (6.19) defining Θ_μ in terms of certain integrals that are critically dependent on the angle of incidence, θ_0, although an inspection of (6.19) reveals that the phase shift on reflection is now an extremely complex function of incidence angle rather than the constant value(s) associated with normal incidence. What is envisaged as useful in future applications involving the prediction of the free modes and resonance frequencies of finite panels under localized forcing is a *simple* functional relation expressing the reflection coefficient in terms of the angle of incidence θ_0; in view of the complicated nature of the Wiener–Hopf kernel it would perhaps be highly optimistic to expect a simple expression to emerge from the analysis. However, the numerical plots, as presented in figures 1 and 2 *do* display significant general



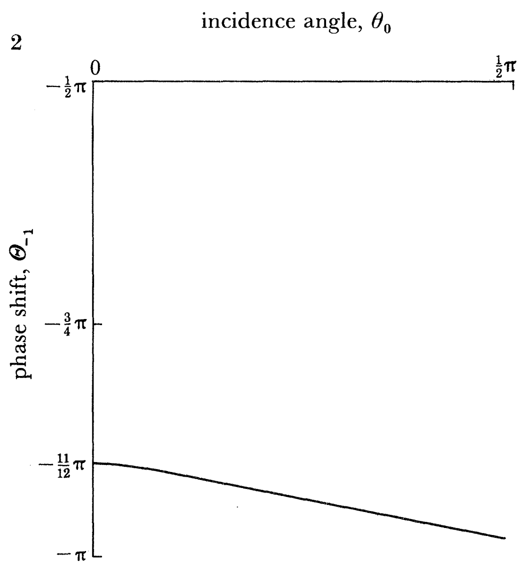


FIGURE 1. Numerical plot for oblique incidence, unbaffled membrane, free edge. Phase shift calculated from (6.19).

FIGURE 2. Numerical plot for oblique incidence, unbaffled membrane, fixed edge. Phase shift calculated from (6.19).

trends, particularly for the reflection at a fixed edge, when the phase angle and angle of incidence appear to be almost linearly related over the whole range, $0 \leqslant \theta_0 \leqslant \frac{1}{2}\pi$. Finally, it is not felt that the general result – that the incident wave is reflected without change in magnitude, but merely suffers a phase shift on reflection – is in any way special to the combination of configuration and edge constraints considered, but that it is an outcome of the assumed heavy fluid-loading limit and, in particular, a manifestation of the virtual fluid incompressibility in this limit. Thus we feel that the cases involving membranes and simple edge constraints may be regarded, in some sense, as being typical and as such they are useful

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in circumstances where the solutions for more realistic physical situations are at present unavailable.

7. Extensions of the approximate method for heavy fluid loading to geometries involving non-normal incidence: strips and rectangular panels under point forcing

The extension of the approximate method of §5 to finite panels subject to localized drive is of great practical concern. One such configuration, namely the circular plate with eccentric drive, was dealt with in a previous section. A simple treatment was successful for that particular problem because the circular panel is (like the strip with line drive, but unlike the square panel with point drive) essentially a problem of normal incidence and locally planar wavefronts. However, in geometries that involve straight edges meeting at an angle the simple approach of the previous chapter is of little use because waves are now incident on a given edge *from all angles* and the reflection coefficient is no longer a constant even at substantial levels of fluid loading. Two typical problems involving the oblique incidence of a surface wave on the edge of a semi-infinite membrane have been examined in some detail in §6. In the low frequency limit it was shown that, when the surface is unbaffled and the edge either fixed or free, the reflection coefficient can be written as a phase shift which is a function of angle of incidence alone. We now describe how that result may be used to obtain the solutions to realistic problems involving finite strips under point drive and, more importantly, finite rectangular panels subject to localized eccentric forcing and with arbitrary edge constraints. For reasons of simplicity, which will soon emerge, we confine our attention to cases in which the structural component of the coupled system is a membrane; the application of the method to thin elastic plates will be mentioned later.

Calculation of the acoustic field, unless all the dimensions are small compared with the acoustic wavelength, is not simple and we confine our discussion to the surface response alone.

7.1. *Illustration from vacuum dynamics*

First, we consider the '*in vacuo*' problem: a homogeneous elastic membrane of tension T and specific surface mass m, lies at rest in equilibrium in the region $|x_1| < a$, $|x_2| < b$ of the (x_1, x_2) plane, and the edges of the membrane are fixed. The system is excited by the time harmonic point force, $F_0\,\delta(x_1)\,\delta(x_2)\,\hat{x}_3\,\mathrm{e}^{-\mathrm{i}\omega t}$ $(\omega > 0)$, which is located at the origin and acts in the positive direction of the x_3-axis. If the velocity normal to the surface in the positive x_3-direction is denoted by $V(x_1, x_2)$, then the time reduced problem requires the solution of

$$(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + k_\mathrm{m}^2)\,(V/-\mathrm{i}\omega) = (F_0/T)\,\delta(x_1)\,\delta(x_2), \tag{7.1}$$

subject to $V = 0$ on

$$\left.\begin{aligned} |x_1| &= a, \quad |x_2| < b, \\ |x_2| &= b, \quad |x_1| < a. \end{aligned}\right\} \tag{7.2}$$

Here, as usual, $k_\mathrm{m} = (m\omega^2/T)^{\frac{1}{2}}$ is the vacuum free wavenumber on the membrane at frequency ω.

In precisely the same way as in §5, it is argued that the surface response comprises two distinct parts: first, a term associated with the response of an infinite membrane under point drive, this

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totally incorporating the singularity due to the forcing; and second, a correction field to allow for the presence of travelling surface waves arising in the reflection process at the edges.

But the response of an infinite membrane to the point force F_0 located at the origin is easily obtained by Fourier transform methods. Indeed,

$$V_\infty(x_1, x_2) = -\frac{\mathrm{i}\omega F_0}{(2\pi)^2 T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k_1|x_1|+\mathrm{i}k_2|x_2|)\,\mathrm{d}k_1\,\mathrm{d}k_2}{(k_1^2+k_2^2-k_\mathrm{m}^2)}, \tag{7.3}$$

where the usual convention applies to the path of integration in each integral. For fixed values of k_2, the single pole at $k_1 = +(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}$ is picked up when the k_1-contour is closed in the upper half-plane. (This pole is either real or purely imaginary according as $k_2 \lessgtr k_\mathrm{m}$.) It follows from (7.3) that

$$V_\infty(x_1, x_2) = \frac{\omega F_0}{4\pi T}\int_{-\infty}^{\infty}\frac{\exp[\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}|x_1|+\mathrm{i}k_2|x_2|]\,\mathrm{d}k_2}{(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}}. \tag{7.4}$$

Since the wavenumber components of the travelling surface modes associated with the unforced system must be such that $k_1^2+k_2^2 = k_\mathrm{m}^2$, it is possible to introduce the field V_c (caused by the edges at $|x_1| = a$) in the form

$$V_\mathrm{c} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k_1|x_1|+\mathrm{i}k_2|x_2|)\,\delta(k_1^2+k_2^2-k_\mathrm{m}^2)\mathrm{d}k_1\,\mathrm{d}k_2}{G(k_1, k_2)}, \tag{7.5}$$

where $\delta(k)$ is the Dirac delta function (Lighthill 1958, p. 10) and where $G(k_1, k_2)$ is an, as yet, unknown function – analogous to the constant of equation (6.4) of Crighton & Innes (1983) – which is subsequently fixed by satisfaction of the phase shift criterion relevant to the imposed edge constraint. Equation (7.5) represents a solution to the homogeneous problem with even symmetry in x_1 and x_2, comprising all the waves with total wavenumber $(k_1^2+k_2^2)^{\frac{1}{2}}$ equal to the free wavenumber k_m.

The replacement of $\delta(k_1^2+k_2^2-k_\mathrm{m}^2)$ by

$$[\delta(k_1-(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}})+\delta(k_1+(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}})]/2(k_\mathrm{m}^2-k_2)^{\frac{1}{2}}$$

in (7.5) allows evaluation of the k_1-integral:

$$V_\mathrm{c} = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}k_2|x_2|}\exp(\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}|x_1|)}{2(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}G((k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}, k_2)}\mathrm{d}k_2+\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}k_2|x_2|}\exp(-\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}|x_1|)}{2(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}G((k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}, k_2)}\mathrm{d}k_2, \tag{7.6}$$

where we have exploited the fact that

$$G(-k_1, k_2) = G(k_1, k_2). \tag{7.7}$$

(This property follows from the assumed symmetry in an even-mode solution, though the method may be easily modified to deal with antisymmetric modes.)

It is now possible to write the sum $V_\mathrm{s} = V_\infty + V_\mathrm{c}$ in a form in which the waves incident and reflected at the edge $|x_1| = a$ are easily identified. Specifically,

$$V_\mathrm{s} = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}k_2|x_2|}\,\mathrm{e}^{\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}a}}{2(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}}\left[\frac{\omega F_0}{2\pi T}+\frac{1}{G((k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}, k_2)}\right]\exp[\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}(|x_1|-a)]\,\mathrm{d}k_2$$
$$+\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}k_2|x_2|}\,\mathrm{e}^{-\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}a}}{2(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}}\left[\frac{1}{G((k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}, k_2)}\right]\exp[-\mathrm{i}(k_\mathrm{m}^2-k_2^2)^{\frac{1}{2}}(|x_1|-a)]\,\mathrm{d}k_2. \tag{7.8}$$

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Then it is argued that for fixed values of k_2 the relationship between the terms in $\exp[\pm i(k_m^2-k_2^2)^{\frac{1}{2}}(|x_1|-a)]$ is that which holds in the corresponding semi-infinite problem with a fixed edge at $x_1 = a$.

In the presence of fluid loading this is only possible if the plate size is large (in some sense which can be quantified) so that the coupling with distant boundaries through the body of the fluid is weak. For the specific case of membrane in a vacuum the relationship is exact because the effect of the boundaries is simply included in the propagating waves whose reflection is being discussed. It could not be exact for an elastic plate '*in vacuo*' because of the coupling via evanescent modes, but the coupling would be weak for a plate whose size exceeds a vacuum plate wavelength.

Consider an identical unloaded semi-infinite membrane and let the wave

$$\eta_i = d\exp[ik_m x_1\cos\theta + ik_m x_2\sin\theta] \tag{7.9}$$

be incident from $x_1 = -\infty$ on the fixed edge located at $x_1 = 0$. Then the total deflection is

$$\eta = d\exp[ik_m x_1\cos\theta + ik_m x_2\sin\theta] + Rd\exp[-ik_m x_1\cos\theta + ik_m x_2\sin\theta], \tag{7.10}$$

and the imposed constraint $\eta(0) = 0$ fixes the reflection coefficient as

$$R = -1. \tag{7.11}$$

Using this in (7.8) determines the unknown function

$$G((k_m^2-k_2^2)^{\frac{1}{2}}, k_2) = -(\omega F_0/4\pi T)\,e^{i(k_m^2-k_2^2)^{\frac{1}{2}}a}/\cos[(k_m^2-k_2^2)^{\frac{1}{2}}a], \tag{7.12}$$

and hence

$$V = -\frac{i\omega F_0}{4\pi T}\int_{-\infty}^{\infty}\frac{e^{ik_2|x_2|}\sin[(k_m^2-k_2^2)^{\frac{1}{2}}(a-|x_1|)]}{(k_m^2-k_2^2)^{\frac{1}{2}}\cos[(k_m^2-k_2^2)^{\frac{1}{2}}a]}\,dk_2, \tag{7.13}$$

a result we expect to be exact for the membrane in vacuum.

Although (7.13) is the solution of the governing differential equation for the motion of the membrane which vanishes along $|x_1| = a$, it does not (and cannot) satisfy the remaining boundary condition along the edges parallel to the x_1-axis. Hence, V_s may be simply identified with the response of an unloaded strip membrane, infinite in length, but of finite width $2a$, excited by a time harmonic point force acting normal to the surface and located at the origin. Evaluation of the integral involved by the calculus of residues, leads to a more convenient form in terms of a sum of normal modes. On closing the contour of integration in the upper half-plane, it follows that

$$V_s = -\frac{\omega F_0}{2Ta}\sum_{p=0}^{\infty} e^{i\kappa_p|x_2|}\cos[(\pi x_1/a)(p+\tfrac{1}{2})]/\kappa_p, \tag{7.14}$$

where

$$\kappa_p = +[k_m^2-(p+\tfrac{1}{2})^2\pi^2/a^2]^{\frac{1}{2}}, \tag{7.15}$$

and κ_p is real or purely imaginary according as $k_m \gtrless (p+\frac{1}{2})\pi/a$.

A solution to (7.1) which *does* satisfy the edge constraint along $|x_2| = b$ is easily constructed by adding to (7.14) a sum of suitably chosen eigenfunctions.

Thus we let

$$V = -\frac{\omega F_0}{2Ta}\sum_{p=0}^{\infty} e^{i\kappa_p|x_2|}\cos[(\pi x_1/a)(p+\tfrac{1}{2})]/\kappa_p + \sum_{p=0}^{\infty} v_p\cos(\kappa_p x_2)\cos[(\pi x_1/a)(p+\tfrac{1}{2})], \tag{7.16}$$

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where each coefficient v_p is fixed so that the individual modes satisfy the phase shift criterion at $|x_1| = b$. Separate modes have reflection coefficient -1, in accordance with (7.11), if

$$v_p = (\omega F_0/2Ta)\,\mathrm{e}^{\mathrm{i}\kappa_p b}/\kappa_p \cos(\kappa_p b). \tag{7.17}$$

Thus it finally emerges that

$$V(x_1, x_2) = \frac{\mathrm{i}\omega F_0}{2Ta} \sum_{p=0}^{\infty} \left\{ \frac{\cos[(\pi x_1/a)(p+\frac{1}{2})]\sin[\kappa_p(b-|x_2|)]}{\kappa_p \cos(\kappa_p b)} \right\}. \tag{7.18}$$

However, for simple uncoupled problems such as this there are many well established methods of solution. One usual approach is to expand the unknown quantity $V(x_1, x_2)$ as a double Fourier sum $\Sigma_{p=0}^{\infty}\Sigma_{q=0}^{\infty} v_{p,q}\cos[(q+\frac{1}{2})\pi x_2/b]\cos[(p+\frac{1}{2})\pi x_1/a]$ and this rapidly leads to the familiar result

$$V(x_1, x_2) = +\frac{\mathrm{i}\omega F_0}{T}\frac{1}{(ab)} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\cos[(q+\frac{1}{2})\pi x_2/b]\cos[(p+\frac{1}{2})\pi x_1/a]}{(p+\frac{1}{2})^2\pi^2/a^2 + (q+\frac{1}{2})^2\pi^2/b^2 - k_{\mathrm{m}}^2}. \tag{7.19}$$

The two expressions (7.18) and (7.19) are easily reconciled by re-writing the latter as

$$V(x_1, x_2) = +\frac{\mathrm{i}\omega F_0}{T}\frac{1}{(ab)} \sum_{p=0}^{\infty} \mathrm{S}_p \cos[(p+\tfrac{1}{2})\pi|x_1|/a], \tag{7.20}$$

where

$$\mathrm{S}_p = \sum_{q=0}^{\infty} \frac{\cos[(q+\frac{1}{2})\pi|x_2|/b]}{(q+\frac{1}{2})^2\pi^2/b^2 - \kappa_p^2}, \tag{7.21}$$

with κ_p as previously defined by (7.15). Series of this type are easily summed by contour integration (Titchmarsh 1947). In our particular case we consider $\oint_{\mathrm{C}} f(z)\,\mathrm{d}z$ where

$$f(z) = \frac{\pi \operatorname{cosec}\pi z \cos[(z\pi/b)(|x_2|-b) + \pi|x_2|/2b]}{(z+\frac{1}{2})^2\pi^2/b^2 - \kappa_p^2}, \tag{7.22}$$

and C is the circle $z = R\,\mathrm{e}^{\mathrm{i}\theta}$, R large. Now on the circle it can be shown that

$$|f(z)| \sim R^{-1}\exp[-\pi R|x_2 \sin\theta|/2b].$$

Also $f(z)$ has simple poles within the contour at the points $z = q$, $(q = 0, \pm 1, \pm 2, \dots)$ and $z = -\frac{1}{2} \pm b\kappa_p/\pi$. Thus, letting $R \to \infty$, we get

$$\mathrm{S}_p = (b/2\kappa_p)\sin[\kappa_p(b-|x_2|)]/\cos(\kappa_p b), \tag{7.23}$$

and on substitution into (7.20) we obtain

$$V(x_1, x_2) = \frac{\mathrm{i}\omega F_0}{2Ta} \sum_{p=0}^{\infty} \frac{\sin[\kappa_p(b-|x_2|)]\cos[(p+\frac{1}{2})\pi x_1/a]}{\kappa_p \cos(\kappa_p b)}. \tag{7.24}$$

This is in precise agreement with (7.18). However, we do not advocate adoption of the approach outlined here to simple cases where elementary methods suffice; rather we emphasize that the present method is particularly useful for coupled problems involving heavy fluid loading where conventional methods fail and where, at the moment, there is no hope of a rigorous treatment.

7.2. *Fluid loaded strip and rectangular panel*

To validate this claim we consider an identical unbaffled rectangular membrane, but now *totally* immersed in a compressible fluid of density ρ and sound speed c_0. The edges of the panel

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are either fixed or free (or indeed any combination of the two if one is prepared to forgo convenient symmetry). The infinite membrane response is easily obtained by transform methods and we find that, in general,

$$V_\infty = -\frac{i\omega F_0}{T}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(ik_1|x_1|+ik_2|x_2|)\,\gamma\,dk_1\,dk_2}{(k_1^2+k_2^2-k_m^2)\,\gamma-\mu k_m^2}, \tag{7.25}$$

where γ is the ubiquitous acoustic square root function. In the heavy fluid-loading limit we recall that, to leading order at any rate, the fluid is incompressible and it is appropriate to replace γ by $(k_1^2+k_2^2)^{\frac{1}{2}}$. Hence

$$V_\infty \sim -\frac{i\omega F_0}{T}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(ik_1|x_1|+ik_2|x_2|)\,(k_1^2+k_2^2)^{\frac{1}{2}}}{(k_1^2+k_2^2)^{\frac{3}{2}}-\mu k_m^2}\,dk_1\,dk_2. \tag{7.26}$$

In the low frequency limit we write $\kappa = (\mu k_m^2)^{\frac{1}{3}}$ so that κ is the free subsonic wavenumber with fluid loading taken into account. Then, for fixed values of k_2 and if $\kappa|x_1| \gg 1$, the dominant term arises from the pole at $k_1 = +(\kappa^2-k_2^2)^{\frac{1}{2}}$. Thus

$$V_\infty \sim \frac{\omega F_0}{6\pi T}\int_{-\infty}^{\infty}\frac{e^{ik_2|x_2|}\exp(i(\kappa^2-k_2^2)^{\frac{1}{2}}|x_1|)}{(\kappa^2-k_2^2)^{\frac{1}{2}}}\,dk_2. \tag{7.27}$$

By analogy with (7.5) in the assumed limit we take

$$V_c = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(ik_1|x_1|+ik_2|x_2|)\,\delta(k_1^2+k_2^2-\kappa^2)}{G(k_1,k_2)}\,dk_1\,dk_2, \tag{7.28}$$

with $G(k_1, k_2)$ to be determined; once again we assume that G is even in both arguments and thence we obtain

$$V_c = \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ik_2|x_2|}\{\exp[i(\kappa^2-k_2^2)^{\frac{1}{2}}|x_1|]+\exp[-i(\kappa^2-k_2^2)^{\frac{1}{2}}|x_1|]\}}{(\kappa^2-k_2^2)^{\frac{1}{2}}G((\kappa^2-k_2^2)^{\frac{1}{2}},k_2)}\,dk_2. \tag{7.29}$$

Near $|x_1| = a$,

$$V_s = V_\infty + V_c \sim \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ik_2|x_2|}\,e^{i(\kappa^2-k_2^2)^{\frac{1}{2}}a}}{(\kappa^2-k_2^2)^{\frac{1}{2}}}\left[\frac{\omega F_0}{3\pi T}+\frac{1}{G((\kappa^2-k_2^2)^{\frac{1}{2}},k_2)}\right]\exp\{i(\kappa^2-k_2^2)^{\frac{1}{2}}(|x_1|-a)\}\,dk_2$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ik_2|x_2|}\,e^{-i(\kappa^2-k_2^2)^{\frac{1}{2}}a}}{(\kappa^2-k_2^2)^{\frac{1}{2}}}\left[\frac{1}{G((\kappa^2-k_2^2)^{\frac{1}{2}},k_2)}\right]\exp\{-i(\kappa^2-k_2^2)^{\frac{1}{2}}(|x_1|-a)\}\,dk_2. \tag{7.30}$$

Now we choose $G((\kappa^2-k_2^2)^{\frac{1}{2}}, k_2)$ so that the phase shift on reflection at $|x_1| = a$ is $\Theta(k_2)$ where the functional form of Θ depends on the imposed edge constraint and in a given problem should be chosen accordingly. In any case, when this criterion is satisfied V_s reduces to

$$V_s = \frac{i\omega F_0}{6\pi T}\int_{-\infty}^{\infty}\frac{e^{ik_2|x_2|}\cos\{(\kappa^2-k_2^2)^{\frac{1}{2}}(a-|x_1|)+\frac{1}{2}\Theta(k_2)\}}{(\kappa^2-k_2^2)^{\frac{1}{2}}\sin\{(\kappa^2-k_2^2)^{\frac{1}{2}}a+\frac{1}{2}\Theta(k_2)\}}\,dk_2. \tag{7.31}$$

Further, if $\lambda_{p,j}$ $(j = 1, 2, \ldots)$ denotes the jth zero in R_+ of the transcendental equation

$$(\kappa^2-k_2^2)^{\frac{1}{2}}a+\tfrac{1}{2}\Theta(k_2) = p\pi, \quad (p = 0, \pm 1, \ldots), \tag{7.32}$$

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then it follows that

$$V_s \sim \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} v_{p,j} \exp(\mathrm{i}\lambda_{p,j}|x_2|) \cos[(|x_1|/a)(p\pi - \tfrac{1}{2}\Theta(\lambda_{p,j}))], \tag{7.33}$$

where

$$v_{p,j} = -(\omega F_0/3T)[\tfrac{1}{2}(\kappa^2 - \lambda_{p,j}^2)\Theta'(\lambda_{p,j}) - a\lambda_{p,j}]^{-1}, \tag{7.34}$$

and where the prime signifies differentiation with respect to the argument.

For general functions $\Theta(k_2)$ the sum over j in (7.33) is infinite. However, when $\Theta(k_2)$ takes simple forms, the index may take on a finite number of values only. For example, when $\Theta(k_2)$ is constant there is a single relevant root of (7.32) for each value of p, and (7.33) then assumes a form similar to the vacuum result; we shall return to this later.

We remark at this point that (7.33) represents the field generated by the point force acting on an infinite strip membrane, $|x_1| < a$, $-\infty < x_2 < \infty$; the field (7.33) satisfies the required conditions on $x_1 = \pm a$, has the appropriate singularity for the point force, and has outgoing wave behaviour as $x_2 \to \pm\infty$.

Returning to the finite rectangular panel, the boundary condition at $|x_2| = b$ may be accommodated by adding to V_s the double sum

$$\sum_{p=0}^{\infty} \sum_{j=1}^{\infty} E_{p,j} \cos[\lambda_{p,j} x_2] \cos[(x_1/a)(p\pi - \tfrac{1}{2}\Theta(\lambda_{p,j}))],$$

and then satisfying the phase shift criterion at $x_2 = b$. This is a straightforward matter and yields the following expression for the surface response:

$$V(x_1, x_2) \sim -\frac{\mathrm{i}\omega F}{3T} \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \cos\left[\frac{x_1}{a}(p\pi - \tfrac{1}{2}\Theta(\lambda_{p,j}))\right] \times \frac{\cos[\lambda_{p,j}(b - |x_2|) + \tfrac{1}{2}\Theta(\lambda_{p,j})]}{[\tfrac{1}{2}(\kappa^2 - \lambda_{p,j}^2)\Theta(\lambda_{p,j}) - a\lambda_{p,j}]\sin[b\lambda_{p,j} + \tfrac{1}{2}\Theta(\lambda_{p,j})]}. \tag{7.35}$$

It is not felt worthwhile to pursue the general solution further; instead, we focus on the specific case alluded to earlier: that of the unbaffled membrane with fixed edges. It has been observed from figure 2 that in this case the phase shift is fairly constant over the whole range of incidence angle, θ_0.

An inspection of (7.22) for $\lambda_{p,j}$ shows that when $\Theta = \Theta_0$, a constant, there is a single root in R_+, corresponding to each value of p, given by

$$\lambda_p = +[\kappa^2 - (p - (1/2\pi)\Theta_0)^2 \pi^2/a^2]^{\frac{1}{2}}. \tag{7.36}$$

Thus a crude first approximation to the normal velocity of the panel is

$$V(x_1, x_2) \sim \frac{1}{3a}\frac{\mathrm{i}\omega F_0}{T} \sum_{p=0}^{\infty} \left\{\frac{\cos[(\pi x_1/a)(p - (1/2\pi)\Theta_0)]\cos[\lambda_p(b - |x_2|) + \tfrac{1}{2}\Theta_0]}{\lambda_p \sin[\lambda_p b + \tfrac{1}{2}\Theta_0]}\right\}. \tag{7.37}$$

By analogy with the vacuum case it may be inferred that this is equivalent to

$$V(x_1, x_2) \sim \frac{2}{3}\frac{1}{ab}\frac{\mathrm{i}\omega F_0}{T} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left\{\frac{\cos\{(\pi x_1/a)[p - (1/2\pi)\Theta_0]\}\cos\{(\pi x_2/b)[q - (1/2\pi)\Theta_0]\}}{(\pi^2/a^2)[p - (1/2\pi)\Theta_0]^2 + (\pi^2/b^2)[q - (1/2\pi)\Theta_0]^2 - \kappa^2}\right\}, \tag{7.38}$$

and it is a straighforward matter to verify rigorously that this is indeed so.

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Equation (7.38) is similar in form to the vacuum result and this is particularly obvious if we recall that, when the edges of the membrane are fixed,

$$\Theta_{-1} \sim -\tfrac{23}{24}\pi. \tag{7.39}$$

Hence

$$V(x_1, x_2) \sim \frac{2}{3}\frac{1}{ab}\frac{\mathrm{i}\omega F_0}{T}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\cos[(\pi|x_1|/a)\,(\overline{p+\frac{1}{2}-\frac{1}{48}})]\cos[(\pi|x_2|/b)\,(\overline{q+\frac{1}{2}-\frac{1}{48}})]}{(\pi^2/a^2)\,(\overline{p+\frac{1}{2}-\frac{1}{48}})^2+(\pi^2/b^2)\,(\overline{q+\frac{1}{2}-\frac{1}{48}})^2-\kappa^2}. \tag{7.40}$$

Therefore the panel has free modes of oscillation (p, q) with mode shapes

$$\cos[(\pi|x_1|/a)\,(\overline{p+\tfrac{1}{2}-\tfrac{1}{48}})]\cos[(\pi|x_2|/b)\,(\overline{q+\tfrac{1}{2}-\tfrac{1}{48}})];$$

the natural frequencies are of the form

$$\omega_{pq} = (T/m)^{\frac{1}{2}}N^{\frac{1}{3}}[(\pi^2/a^2)\,(\overline{p+\tfrac{1}{2}-\tfrac{1}{48}})^2+(\pi^2/b^2)\,(\overline{q+\tfrac{1}{2}-\tfrac{1}{48}})^2]^{\frac{1}{2}}. \tag{7.41}$$

It should, perhaps, be repeated that these results are invalid near the edges, near the corners and in the immediate neighbourhood of the drive point. At distances within $O(\kappa^{-1})$ of the boundary the total field may be taken care of by using the full solution to the semi-infinite Wiener–Hopf problem and that comprises – for frequencies below coincidence at any rate – propagating incident and reflected waves, the latter associated with the residue contribution from the real pole in R_+, waves that decay exponentially away from the membrane associated with complex poles in R_+ which of necessity have a positive imaginary part, and finally a branch line integral that may be expressed as a Fresnel (error) integral (Noble 1958), of a type familiar in all half-plane problems in which the integral contains poles and square root branch points. In the vicinity of the forcing where the wavefronts are not locally planar, an expression of the above type in terms of normal modes is obviously inappropriate. For membranes large on the structural wavelength size (that is, large in the sense that $k_m a, k_m b \gg N^{\frac{1}{3}}$) the complementary functions associated with the edges may be used as they stand, but the infinite membrane contribution should be retained in the full integral form (or equivalent), as was done in §4 for the circular plate. The motion near the corners is not an easy matter to resolve; it requires the solution to the quarter-plane scattering problem and that is unavailable at this moment. If that solution were known, a thorough analysis of the problem could be done on the basis of matched asymptotic expansions.

We also note that, although the mode shapes and natural frequencies are close in form to the vacuum dynamics (in practice $N^{\frac{1}{3}}$ is unlikely to be very much less than unity), the amplitude of the modes with high values of p or q in the loaded and unloaded cases is in the ratio 2:3 for a given F_0.

We remark finally that the expression for the drive admittances for the strip and the rectangular panel are immediately derived from (7.33) and (7.38) respectively. We have not obtained any more succinct representation than the doubly infinite series, but can observe that while the drive admittance for the rectangle is purely imaginary, that for the strip includes a resistive component associated with the modes which can propagate to infinity along the strip.

8. Conclusions

In this paper we have shown that the original strip plate problem of Crighton & Innes (1983) is typical of its class and we have obtained results, analogous to those of §6 of Crighton & Innes, for a wide variety of configurations and edge conditions, each of some practical interest. We

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have also demonstrated that the method can be applied to forced panels with curved boundaries; for example, the circular plate with eccentric drive has been discussed. In both classes of problem we have presented simple expressions for the drive admittances in the heavy fluid loading–low frequency limit.

Furthermore, we have illustrated the fact that the basic concept, i.e. in the heavy fluid loading limit, the structural reflection is characterized by a mere phase shift, remains true even when the incidence upon the edge is oblique. The prototypical semi-infinite Wiener–Hopf problem, needed here as a building block in the study of more complicated finite structures, is correspondingly more difficult, but we have obtained expressions for the phase shift on reflection for two different edge conditions on an unbaffled membrane. These expressions are not particularly transparent and the general trends are more easily identified from numerical plots of phase shift against angle of incidence.

We have then extended the approximate method in a general form suitable for use in two-dimensional planar problems. In passing we have obtained, as a by-product of the method, an expression for the response of a strip under point drive. The veracity of the method in one particular case – the *in vacuo* rectangular membrane – was indicated; for such a simple problem the method produced an exact expression for the surface response in total agreement with that from elementary (that is, Fourier series) analysis.

In the heavy fluid-loading limit we noted that for one case in particular, the unbaffled membrane with a fixed edge, the phase shift is particularly simple. This enabled the natural modes and frequencies to be written down in a form which is very similar to the vacuum dynamics.

In almost all of the model problems considered in this paper we have concentrated on configurations involving symmetric excitation provoking an even-mode response. Similar analyses for sums of odd or general modes are straightforward. In every situation considered in detail we have predicted the existence of 'natural modes' and associated resonances (to leading order at any rate).

The key factor in this assumed limit is the near incompressibility of the fluid, and the occurrence of resonances is essentially linked to the basic 'hydrodynamic' nature of the fluid motion.

In a simplified plate model, involving a surface of constant stiffness, we have been able to verify that the energy lost in the edge-reflection process *exactly* balances the radiated acoustic power. In the incompressible limit this scattered power vanishes and the magnitude of the reflection coefficient is unity.

For the general coupled plate–fluid complex in the assumed limit the calculated scattered acoustic field is a quite insignificant fraction of the total incident power and this leads to a consequent build-up of structural energy and the occurrence of resonance. This is quite unlike the mechanism in light fluid-loading problems where the effect of the fluid is negligible and to leading order its presence may be ignored.

It would be possible to attempt to determine the natural modes and frequencies of heavily fluid-loaded membranes for configurations in which the phase shift $\Theta(\theta_0)$ is rather more complicated than the constant value attained in the specific problem examined in detail here, though this is unlikely to produce *simple* results. We prefer at present to consider the equivalent problem in which the membrane is replaced by a plate. A general low frequency solution, analogous to (7.35) in the form of a double sum is easily written down, but the progress one

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can make from that point is limited by the form of the phase shift function $\Theta(\theta_0)$. The technical difficulties introduced lie in the Wiener–Hopf split of the dispersion function which now involves (in addition to θ_0) the Poisson ratio, ν, of the plate material. This is introduced via the assumed edge conditions (see Rayleigh 1945, p. 372). It would be highly optimistic to expect to achieve a Wiener–Hopf split for general values of ν, but it is hoped that simple results might be obtained in some assumed $\nu \ll 1$ limit, or indeed there may possibly be gross simplifications associated with specific parameter values. These studies are promised in the near future.

In conclusion then, we feel that we have made real progress in the understanding of the response of finite panels to localized forcing. An approximate method has been devised which may be used in heavy fluid-loading problems for which no rigorous approach is possible, and, moreover, the simple concepts on which the method is based may, we hope, be applied in conjunction with finite element methods to extremely complicated structures for which advance is otherwise impossible (if the effects of fluid loading are to be retained at all).

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Appendixes

A.1. *The factorization of the Wiener–Hopf kernel for the thin plate; an exact factorization and low frequency asymptotics*

In this Appendix we first present various factorizations of the Wiener–Hopf kernel,

$$K(k) = (k^4-k_p^4)\,(k^2-k_0^2)^{\frac{1}{2}}-\mu k_p^4, \tag{A 1}$$

into factors $K_\pm(k)$ analytic and non-zero in overlapping regions $R_\pm$ of the complex wavenumber plane and such that

$$K(k) = K_+(k)\,K_-(k). \tag{A 2}$$

An exact factorization (valid for all parameter ranges) is obtainable, but it is not particularly illuminating and in specific limits alternative approaches involving approximation at an earlier stage are certainly speedier and easier to handle. In many cases the leading term of $K_+(k)$ is all that is required and therefore we begin by considering an approximate method for producing leading order terms in the heavy fluid-loading limit specifically.

(*a*) *An approximate factorization*

Introduce the new variables $k = k_p z$, $k_0 = Mk_p$, $\mu/k_p = \sigma^5/M$ into (A 1); then we have

$$K(k) = (k_p^5/M)\,\mathcal{K}(z), \tag{A 3}$$

where

$$\mathcal{K}(z) = M(z^4-1)\,(z^2-M^2)^{\frac{1}{2}}-\sigma^5. \tag{A 4}$$

Our usual characterization of the heavy fluid-loading limit (see §1 for the precise definition) is equivalent to considering (A 4) with $\sigma = O(1)$ and $M \ll 1$. At frequencies well below the 'coincidence' frequency ($M = 1$) it can be shown that an approximation of $\mathcal{K}(z)$, uniformly valid in z, is given by

$$\mathcal{K}(z) \sim Mz^4(z^2)^{\frac{1}{2}}-\sigma^5, \tag{A 5}$$

where $(z^2)^{\frac{1}{2}}$ is to be interpreted as $|z|$ on the real axis and may then be continued throughout the entire z-plane, cut from 0 to $\pm i\infty$ in $R_\pm$ respectively, as $z\,\mathrm{sgn}\,\mathrm{Re}\,z$. The approximation of the kernel given by (A 5) seems reasonable on physical grounds for the following reasons. For $M \ll 1$ the fluid motion is almost incompressible and hence $(z^2-M^2)^{\frac{1}{2}}$ may be replaced by $(z^2)^{\frac{1}{2}}$ unless $z = O(M)$ when, in any case, the first term in (A 4) is negligible. For $z = O(1)$ the fluid loading dominates the surface response and only when $z = O(\sigma/M^{\frac{1}{5}})$ are the elastic forces large enough to balance the surface inertia terms and then it is appropriate to replace (z^4-1) by z^4 only.

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For convenience we now introduce the new variable $t = M^{\frac{1}{5}}z$ in (A 5); then logarithmic differentiation of $\mathscr{K}(t) = t^4(t^2)^{\frac{1}{2}} - \sigma^5$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left\{\frac{\mathscr{K}_+(t)\,\mathscr{K}_-(t)}{(t^{10}-\sigma^{10})^{\frac{1}{2}}}\right\}\right] = \frac{5\sigma^5 t^3 (t^2)^{\frac{1}{2}}}{(t^{10}-\sigma^{10})}. \tag{A 6}$$

Now let the zeros of the denominator $(t^{10}-\sigma^{10})$ be denoted by $\pm t_n$ $(n = 1, 2, 3, 4, 5)$; that is,

$$t_n = \sigma \exp \tfrac{1}{5}(n-1)\,\pi \mathrm{i}. \tag{A 7}$$

We note also that the real zero, $t_1 = \sigma$, corresponds to the unique subsonic wavenumber $\kappa = k_p\,\sigma/M^{\frac{1}{5}}$ in the heavy fluid-loading limit. Furthermore, if we introduce quantities α_n such that

$$\alpha_n^{-1} = 10 t_n^9 \quad (n = 1, 2, 3, 4, 5), \tag{A 8}$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left\{\frac{\mathscr{K}_+(t)\,\mathscr{K}_-(t)}{\prod\limits_{1}^{5}(t^2-t_n^2)^{\frac{1}{2}}}\right\}\right] = 5\sigma^5 (t^2)^{\frac{1}{2}} \sum_{n=1}^{5} \alpha_n t_n^3 \left\{\frac{1}{t-t_n} + \frac{1}{t+t_n}\right\}. \tag{A 9}$$

This type of procedure for determining Wiener–Hopf factorizations is standard (see, for example, Crighton & Leppington 1970 or Cannell 1976) and continues as follows. The additive split functions of $(t^2)^{\frac{1}{2}}$ are known (Crighton 1976) to be given by

$$(t^2)^{\frac{1}{2}} = P_+(t) + P_-(t), \tag{A 10}$$

where

$$P_\pm(t) = \tfrac{1}{2}t \pm \tfrac{1}{2}\mathrm{i}t \ln_\pm t, \tag{A 11}$$

and with the branch cuts for the logarithm running from 0 to $\mp\mathrm{i}\infty$ respectively. Moreover, with these cuts the functions $P_\pm(t)$ have the property that

$$P_+(-t) = P_-(t). \tag{A 12}$$

These definitions having been used in (A 9), manipulation of a typical kind allows the right side of (A 9) to be written as a sum of 'plus' and 'minus' functions. Concentrating on the 'plus' function alone leads, after a single quadrature, to

$$\mathscr{K}_+(t) = A_+ \prod_{n=1}^{5}(t+t_n)^{\frac{1}{2}} \exp\left\{-\frac{\mathrm{i}\sigma^5}{\pi}\sum_{n=1}^{5}\left[\frac{1}{t_n^4}\int_t^\infty \frac{\ln_+ u\,\mathrm{d}u}{u^2-t_n^2} - \frac{\ln_+ t_n}{t_n^4}\int_t^\infty \frac{\mathrm{d}u}{u^2-t_n^2}\right]\right\}, \tag{A 13}$$

with a similar expression for $\mathscr{K}_-(t)$. The constant A_+ is chosen, as is often the case, in fulfilment of the convenient requirement $\mathscr{K}_+(-t) = \mathscr{K}_-(t)$ and thus

$$A_+ = \mathrm{e}^{-\frac{5}{4}\pi\mathrm{i}}. \tag{A 14}$$

Hence, from (A 13),

$$\mathscr{K}_\pm(t) \sim -\mathrm{e}^{\mp\frac{1}{4}\pi\mathrm{i}}\, t^{\frac{5}{2}} \quad \text{as} \quad |t| \to \infty \text{ in } R_\pm. \tag{A 15}$$

The integrals in (A 13) may be evaluated in terms of complex dilogarithms, but such expressions are of marginal practical use; alternatively we consider the consequence of (A 13) in a variety of interesting cases.

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(i) $M \ll 1$, $k = O(k_p)$

$$\mathscr{K}_+(t) \sim \sigma^{\frac{5}{2}}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\exp\left\{-\frac{\mathrm{i}\sigma^5}{\pi}\sum_{n=1}^{5}\left[\frac{1}{t_n^4}\int_0^\infty \frac{\ln_+ u\,\mathrm{d}u}{u^2-t_n^2}-\frac{\ln_+ t_n}{t_n^4}\int_0^\infty \frac{\mathrm{d}u}{u^2-t_n^2}\right]\right\} = \sigma^{\frac{5}{2}}\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi}. \quad \text{(A 16)}$$

Hence, $\mathscr{K}_+(t)\,\mathscr{K}_-(t) \sim -\sigma^5$ as $M \to 0$ for $t = O(M^{\frac{1}{5}})$, as expected from, say, (A 5). We note that (A 16) is also valid for $k = O(k_0)$ and hence

$$\left.\begin{aligned} K_+(k_0) &\sim (k_p^{\frac{5}{2}}\,\sigma^{\frac{5}{2}}/M^{\frac{1}{2}})\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi},\\ K_+(\lambda k_p) &\sim (k_p^{\frac{5}{2}}\,\sigma^{\frac{5}{2}}/M^{\frac{1}{2}})\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi}, \quad \lambda = O(1).\end{aligned}\right\} \quad \text{(A 17)}$$

(ii) $M \ll 1$, $k = \kappa (= k_p\sigma/M^{\frac{1}{5}})$

Since the wavenumber $k = \kappa$ corresponds to $t = \sigma$ then

$$\mathscr{K}_+(\sigma) \sim -\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\,\sigma^{\frac{5}{2}}q, \quad \text{(A 18)}$$

with

$$q = \prod_{n=1}^{5}(1+t_n/\sigma)^{\frac{1}{2}}\exp\left\{-\frac{\mathrm{i}\sigma^5}{\pi}\sum_{n=1}^{5}\left[\frac{1}{t_n^4}\int_\sigma^\infty \frac{\ln_+ u\,\mathrm{d}u}{u^2-t_n^2}-\frac{\ln_+ t_n}{t_n^4}\int_\sigma^\infty \frac{\mathrm{d}u}{u^2-t_n^2}\right]\right\}; \quad \text{(A 19)}$$

the evaluation of q is of some interest. First, we deform the contours of integration (for a fixed n) on to the arc of the circle $u = \sigma\mathrm{e}^{\mathrm{i}\theta}$ $(0 \leqslant \theta \leqslant \frac{1}{5}(n-1)\,\pi$, and the ray $u = t_n z$, $(1 \leqslant z \leqslant \infty)$, and close with a circular arc at infinity in R_+. Then

$$\sum_{n=1}^{5}\left[\frac{1}{t_n^4}\int_\sigma^\infty \frac{\ln_+ u\,\mathrm{d}u}{u^2-t_n^2}-\frac{\ln_+ t_n}{t_n^4}\int_\sigma^\infty \frac{\mathrm{d}u}{u^2-t_n^2}\right]$$
$$= \frac{\pi^2}{8\sigma^5}+\frac{\mathrm{i}}{2\sigma^5}\sum_{n=1}^{5}\mathrm{e}^{-(n-1)\,\pi\mathrm{i}}\,[Cl_2(\tfrac{1}{5}(n-1)\,\pi)+Cl_2(\pi-\tfrac{1}{5}(n-1)\,\pi)+\tfrac{1}{5}(n-1)\,\pi\ln\tan\tfrac{1}{10}(n-1)\,\pi], \quad \text{(A 20)}$$

wherein Cl_2 is Clausen's function (Lewin 1981).

The known properties of Cl_2 allow the sum in (A 20) to be evaluated with result $\frac{1}{2}\pi\ln 5$. Thus it follows from (A 19) that

$$q = 10^{\frac{1}{2}}\,\mathrm{e}^{\frac{3}{8}\pi\mathrm{i}}, \quad \text{(A 21)}$$

and finally
$$\mathscr{K}_+(\kappa) \sim -k_p^{\frac{5}{2}}\sigma^{\frac{5}{2}}(10)^{\frac{1}{2}}\,\mathrm{e}^{+\frac{1}{8}\pi\mathrm{i}}/M^{\frac{1}{2}}. \quad \text{(A 22)}$$

In the majority of instances these leading-order approximations suffice. However, when further terms in the asymptotic series are required, exact expressions like (A 13) are a convenient point from which to start.

(*b*) *An exact factorization*

Consider the logarithmic differentiation of

$$\mathscr{K}(z) = \mathscr{K}_+(z)\,\mathscr{K}_-(z) = M(z^2-M^2)^{\frac{1}{2}}(z^4-1)-\sigma^5;$$

after some manipulation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\ln\left\{\frac{\mathscr{K}_+(z)\,\mathscr{K}_-(z)}{[M^2(z^2-M^2)\,(z^4-1)^2-\sigma^{10}]^{\frac{1}{2}}}\right\}\right] = \frac{M\sigma^5\,z(5z^4-4M^2z^2-1)}{[M(z^2-M^2)\,(z^4-1)^2-\sigma^{10}]}\,\frac{1}{(z^2-M^2)^{\frac{1}{2}}}. \quad \text{(A 23)}$$

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Now let the zeros of $M^2(z^2-M^2)(z^4-1)^2-\sigma^{10}$ be denoted by $\pm z_n$, $n = 1, 2, 3, 4, 5$, with $\operatorname{Im} z_n > 0$. Also define

$$\left.\begin{aligned} \alpha_n^{-1} &= \{(\mathrm{d}/\mathrm{d}z)\,[M^2(z^2-M^2)(z^4-1)^2-\sigma^{10}]\}_{z=z_n}, \\ \beta_n &= \alpha_n z_n(5z_n^4-4M^2z_n^2-1), \end{aligned}\right\} \tag{A 24}$$

and introduce the additive split of $(z^2-M^2)^{-\frac{1}{2}}$ (Noble 1958, p. 20) given by

$$(z^2-M^2)^{-\frac{1}{2}} = Q_+(z)+Q_-(z),$$

with

$$Q_\pm(z) = [1/\pi(z^2-M^2)^{\frac{1}{2}}]\arccos(\pm z/M). \tag{A 25}$$

For complex z, $\arccos(z/M)$ should be interpreted as

$$\mathrm{i}\ln\{[z+(z^2-M^2)^{\frac{1}{2}}]/M\}, \quad -\pi \leqslant \operatorname{Im}\ln z < \pi,$$

with the usual (acoustic) branch cuts for $(z^2-M^2)^{\frac{1}{2}}$. Thence, concentrating on the $\oplus$ function alone, we find that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{\mathscr{K}_+(z)}{\prod_{n=1}^5(z+z_n)^{\frac{1}{2}}}\right]\right\} = M\sigma^5\sum_{n=1}^5\frac{\beta_n}{(z-z_n)}[Q_+(z)-Q_+(z_n)] + M\sigma^5\sum_{n=1}^5\frac{\beta_n}{(z+z_n)}[Q_+(z)+Q_+(z_n)], \tag{A 26}$$

where we have used the fact that

$$Q_+(-z) = Q_-(z). \tag{A 27}$$

For $\sigma = O(1)$ and $M \ll 1$ the roots z_n and the quantities α_n, β_n may be calculated to any desired order. For the purposes of §2 it is enough to know that

$$\left.\begin{aligned} z_n &\sim (\sigma/M^{\frac{1}{5}})\exp(\tfrac{1}{5}(n-1)\pi\mathrm{i})\,[1+(M^{\frac{4}{5}}/5\sigma^4)\exp(-\tfrac{4}{5}(n-1)\pi\mathrm{i})], \\ \alpha_n &\sim (1/10\sigma^2M^{\frac{1}{5}})\exp(-\tfrac{9}{5}(n-1)\pi\mathrm{i})\,[1-(3M^{\frac{4}{5}}/5\sigma^4)\exp(-\tfrac{4}{5}(n-1)\pi\mathrm{i})], \\ \beta_n &\sim (1/2\sigma^4M^{\frac{6}{5}})\exp(-\tfrac{4}{5}(n-1)\pi\mathrm{i})\,[1+(M^{\frac{4}{5}}/5\sigma^4)\exp(-\tfrac{4}{5}(n-1)\pi\mathrm{i})]. \end{aligned}\right\} \tag{A 28}$$

(i) $M \ll 1$, $k = O(k_\mathrm{p})$

For $z = O(1)$ the functions $Q_+(z)$ and $Q_+(z_n)$ occurring in (A 26) may be replaced by their power series expansions and we find that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\ln\left\{\frac{\mathscr{K}_+(z)}{\prod_{n=1}^5(z+z_n)^{\frac{1}{2}}}\right\}\right] = a_1M^{\frac{1}{5}}/\sigma+a_3z^2M^{\frac{3}{5}}/\sigma^3+O(M), \tag{A 29}$$

where

$$\left.\begin{aligned} a_1 &= \mathrm{e}^{\frac{1}{10}\pi\mathrm{i}}/2\cos\tfrac{1}{10}\pi+O(M^{\frac{4}{5}}) \\ \text{and}\quad a_3 &= \mathrm{e}^{\frac{3}{10}\pi\mathrm{i}}/2\sin\tfrac{1}{5}\pi+O(M^{\frac{4}{5}}). \end{aligned}\right\} \tag{A 30}$$

Now

$$\sum_{n=1}^5(z+z_n)^{\frac{1}{2}} = -(\sigma^{\frac{5}{2}}/M^{\frac{1}{2}})\{1+b_1M^{\frac{1}{5}}z/2\sigma+(\tfrac{1}{2}b_2-\tfrac{1}{8}b_1^2)M^{\frac{2}{5}}z^2/\sigma^2+(\tfrac{1}{2}b_3-\tfrac{1}{4}b_1b_2+\tfrac{1}{16}b_1^3)M^{\frac{3}{5}}z^3/\sigma^3 + (\tfrac{1}{2}b_4-\tfrac{1}{4}b_1b_3-\tfrac{1}{8}b_2^2+\tfrac{3}{16}b_1^2b_2-\tfrac{5}{128}b_1^4)M^{\frac{4}{5}}z^4/\sigma^4+O(M)\}, \tag{A 31}$$

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where
$$\left.\begin{aligned} b_1 &= -\mathrm{e}^{\frac{3}{5}\pi\mathrm{i}}/\sin\tfrac{1}{10}\pi, \\ b_2 &= \frac{2\cos\frac{1}{5}\pi}{\sin\frac{1}{10}\pi}\,\mathrm{e}^{-\frac{4}{5}\pi\mathrm{i}}, \\ b_3 &= -\frac{2\cos\frac{1}{5}\pi}{\sin\frac{1}{10}\pi}\,\mathrm{e}^{-\frac{1}{5}\pi\mathrm{i}} \\ \text{and}\qquad b_4 &= \mathrm{e}^{\frac{2}{5}\pi\mathrm{i}}/\sin\tfrac{1}{10}\pi. \end{aligned}\right\} \tag{A 32}$$

After some manipulation, term by term integration, exponentiation and a little more manipulation the final outcome is that, for $k = O(k_\mathrm{p})$ and in the low frequency limit,

$$K_+(k) \sim \frac{k_\mathrm{p}^{\frac{5}{2}}\sigma^{\frac{5}{2}}\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi}}{M^{\frac{1}{2}}}\left\{1+\frac{c_1 M^{\frac{1}{5}}k}{k_\mathrm{p}\sigma}+\frac{c_2 M^{\frac{2}{5}}k^2}{k_\mathrm{p}^2\sigma^2}+\frac{c_3 M^{\frac{3}{5}}k^3}{k_\mathrm{p}^3\sigma^3}\right\}, \tag{A 33}$$

where the coefficients are given, to leading order at any rate, by

$$\left.\begin{aligned} c_1 &= -\mathrm{i}\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}/\sin\tfrac{1}{5}\pi, \\ c_2 &= -\frac{\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}}}{2\sin^2\frac{1}{5}\pi}. \end{aligned}\right\} \tag{A 34}$$

The constant A_+ was determined, by comparison with (A 17), to ensure that $K_+(-k) = K_-(k)$.

The asymptotics may be partially checked by using the rigorous method of Kranzer & Radlow (1962). In their notation, $c = -1/\sigma^5$ and $\alpha = 5$; therefore, we have

$$h = \frac{5}{\pi\sigma}\,\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{d}t}{1+t^5} = \frac{\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}}{\sigma\sin\frac{1}{5}\pi}.$$

Allowing for conversion to the upper half plane, and a slight modification in notation, gives

$$\mathscr{K}^*(k)/\mathscr{K}(k) \sim 1-\mathrm{i}\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}\,M^{\frac{1}{5}}k/k_\mathrm{p}\sigma\sin\tfrac{1}{5}\pi,$$

which is in agreement with (A 33) as far as it goes. However, we believe that the direct method presented here is more adaptable than that of Kranzer & Radlow, elegant though that is. We have a straightforward, though, admittedly, sometimes arduous, technique for generating higher-order terms (which Kranzer & Radlow fail to do). The method is also valuable in situations where their exact method is impotent. One instance in which the rigorous approach is ineffective is when $z = O(M^{-\frac{1}{5}})$; the theory of Kranzer & Radlow gives explicit results only for $z = O(1)$.

(ii) $M \ll 1$, $k = \kappa (= k_\mathrm{p}\sigma/M^{\frac{1}{5}})$

Without going into detail we simply state that, using (A 26) as a starting point, an analogous treatment yields

$$K_+(\kappa) \sim -(k_\mathrm{p}^{\frac{5}{2}}\sigma^{\frac{5}{2}}q\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}/M^{\frac{1}{2}})\,[1+\alpha M^{\frac{4}{5}}/5\sigma^4], \tag{A 35}$$

with q as previously defined (by A 21) and

$$\alpha = \tfrac{5}{4}-\mathrm{i}(\cot\tfrac{1}{5}\pi - 3/(10\sin\tfrac{1}{5}\pi)+1/(10\cos\tfrac{1}{10}\pi)). \tag{A 36}$$

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(iii) $M \ll 1$, $|k| \to \infty$

Expressions for the factors of $H_{\pm}(k) = K_{\pm}(k)/\gamma_{\pm}(k)$ as $|k| \to \infty$ in $R_{\pm}$ are also vital. Again it is a pedestrian yet tedious procedure to obtain such expansions. Commencing from (A 26) and formally replacing the quantities involved by the relevant asymptotics, it is a simple matter to obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{\mathscr{K}_+(z)}{\prod_{n=1}^{5}(z+z_n)^{\frac{1}{2}}}\right]\right\} \sim \frac{\mathrm{i}\sigma}{2M^{\frac{1}{5}}}\frac{\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}}\,z^{-2}}{\cos\frac{1}{10}\pi} + \frac{\mathrm{i}\sigma^3\,\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}\,z^{-3}}{2M^{\frac{3}{5}}\sin\frac{1}{5}\pi}, \tag{A 37}$$

where the anticipated logarithmic terms all have coefficient zero, to this order at any rate.

Whence,

$$\frac{\mathscr{K}_+(z)}{A_+\prod_{n=1}^{5}(z+z_n)^{\frac{1}{2}}} \sim 1+\frac{a_1}{z}+\frac{a_2}{z^2}+\frac{a_3}{z^3}, \tag{A 38}$$

with a_1, a_2 and a_3 given by

$$\left.\begin{aligned} a_1 &= -(\mathrm{i}\sigma/2M^{\frac{1}{5}})\,\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}}/\cos\tfrac{1}{10}\pi,\\ a_2 &= -(\sigma^2/8M^{\frac{2}{5}})\,\mathrm{e}^{\frac{4}{5}\pi\mathrm{i}}/\cos^2\tfrac{1}{10}\pi,\\ a_3 &= -(\mathrm{i}\sigma^3\,\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}/6M^{\frac{3}{5}})\,[1/\sin\tfrac{1}{5}\pi+1/8\cos^3\tfrac{1}{10}\pi]. \end{aligned}\right\} \tag{A 39}$$

Now

$$\prod_{1}^{5}(z+z_n)^{\frac{1}{2}} \sim z^{\frac{5}{2}}\left\{1+\frac{b_1}{z}+\frac{b_2}{z^2}+\frac{b_3}{z^3}\ldots\right\}, \quad \text{as } |z|\to\infty \text{ in } R_+, \tag{A 40}$$

where

$$\left.\begin{aligned} b_1 &= (\sigma/M^{\frac{1}{5}})\,\tfrac{1}{2}\mathrm{i}(\mathrm{e}^{-\frac{1}{10}\pi\mathrm{i}}/\sin\tfrac{1}{10}\pi),\\ b_2 &= (\sigma^2/M^{\frac{2}{5}})\,[\mathrm{e}^{\frac{4}{5}\pi\mathrm{i}}\cos\tfrac{1}{5}\pi/\sin\tfrac{1}{10}\pi+\tfrac{1}{8}\mathrm{e}^{-\frac{1}{5}\pi\mathrm{i}}/\sin^2\tfrac{1}{10}\pi],\\ b_3 &= -(\sigma^3/M^{\frac{3}{5}})\,[+\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}\cos\tfrac{1}{5}\pi/\sin\tfrac{1}{10}\pi+\mathrm{i}\mathrm{e}^{-\frac{3}{10}\pi\mathrm{i}}/16\sin^3\tfrac{1}{10}\pi+\tfrac{1}{2}\mathrm{i}\,\mathrm{e}^{\frac{7}{10}\pi\mathrm{i}}\cos\tfrac{1}{5}\pi/\sin^2\tfrac{1}{10}\pi]. \end{aligned}\right\} \tag{A 41}$$

Multiplication of the two series leads to complicated trignometric coefficients that simplify grossly. The upshot is that

$$K_+(k) \sim -\frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,k^{\frac{5}{2}}}{M^{\frac{1}{2}}}\left[1+\frac{k_\mathrm{p}\sigma}{M^{\frac{1}{5}}}\frac{\mathrm{e}^{\frac{3}{10}\pi\mathrm{i}}}{\sin\frac{1}{5}\pi}\frac{1}{k}-\frac{k_\mathrm{p}^2\sigma^2}{M^{\frac{2}{5}}}\frac{\mathrm{e}^{-\frac{2}{5}\pi\mathrm{i}}}{2\sin^2\frac{1}{5}\pi}\left(\frac{1}{k}\right)^2-\frac{k_\mathrm{p}^3\sigma^3\,\mathrm{e}^{-\frac{1}{10}\pi\mathrm{i}}}{M^{\frac{3}{5}}\,6\sin^3\frac{1}{5}\pi}\left(1-\frac{\sin^2\frac{1}{5}\pi}{\cos\frac{1}{5}\pi}\right)\left(\frac{1}{k}\right)^3\right]. \tag{A 42}$$

This almost recovers the parallel result of Cannell (1976) which we believe to be in error in the coefficient of k^{-3}.

Further, since $H_+(k) = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\,K_+(k)/(k+k_0)^{\frac{1}{2}}$, it follows that

$$H_-(k) \sim \mathscr{H}k^2\sum_{n=0}^{3}h_n\,k^{-n}, \tag{A 43}$$

where $h_0 = 1$,

$$\left.\begin{aligned} h_1 &= (k_\mathrm{p}\sigma/M^{\frac{1}{5}})\,\mathrm{e}^{\frac{3}{10}\pi\mathrm{i}}/\sin\tfrac{1}{5}\pi,\\ h_2 &= -(k_\mathrm{p}^2\sigma^2/M^{\frac{2}{5}})\,\mathrm{e}^{-\frac{2}{5}\pi\mathrm{i}}/2\sin^2\tfrac{1}{5}\pi,\\ h_3 &= -(k_\mathrm{p}^3\sigma^3/M^{\frac{3}{5}})\,(\mathrm{e}^{-\frac{1}{10}\pi\mathrm{i}}/6\sin^3\tfrac{1}{5}\pi)\,(1-\sin^2\tfrac{1}{5}\pi/\cos\tfrac{1}{5}\pi)\\ \text{and}\quad \mathscr{H} &= (k_\mathrm{p}^2/M^{\frac{1}{2}})\,\mathrm{e}^{-\pi\mathrm{i}}. \end{aligned}\right\} \tag{A 44}$$

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These are all the asymptotics required in the unbaffled and baffled geometry Wiener–Hopf problems of §§2.2 and 2.3.

A.2. *The determination of the unknown constants arising in the unbaffled geometry Wiener–Hopf problem of* §2.2

The four equations in (2.16) and (2.18) may all be expressed in the form

$$E_0+E_1\lambda-[\omega d/\gamma_\kappa(\lambda+\kappa)]\,[K_-(\lambda)-K_-(-\kappa)]+\mathrm{i}\omega P(\lambda)/K_+(\lambda)=0, \qquad \text{(A 45)}$$

with $\lambda=\pm k_\mathrm{p}$ respectively and where $K_+(-s)$ is to be interpreted as $K(-s)/K_-(-s)$.

We eliminate E_0 and E_1 from these equations by adding and subtracting in pairs; a further addition and subtraction of the resulting equations and use of the asymptotics (A 34) and (A 35) for $K_+(\lambda)$ ($\lambda=\pm k_\mathrm{p}$, $+\mathrm{i}k_\mathrm{p}$) and $K_+(\kappa)$ leads to the following set of equations:

$$\begin{aligned}4E_0&+(\mathrm{i}\omega dk_\mathrm{p}^{\frac{1}{2}}/N^{\frac{1}{10}})[4(q\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}+1)+4(\tfrac{3}{5}(q\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}+1)+\tfrac{1}{5}\alpha q\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\\&+c_1+c_3+c_1c_3+\tfrac{1}{2}c_1^2-\tfrac{1}{8}c_1^4)\,N^{\frac{4}{5}}-(\omega\eta'''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{5}{2}})\,[4+4c_1(c_3-\tfrac{1}{8}c_1^3)\,N^{\frac{4}{5}}]\\&+(\omega\eta''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{3}{2}})\,[-4\mathrm{i}c_3N^{\frac{3}{5}}+(1-\mathrm{i})\,(2c_4+\tfrac{1}{4}c_1^4-c_1^3)\,N^{\frac{4}{5}}]\\&+(\omega\eta'(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{1}{2}})\,[2c_1^2N^{\frac{2}{5}}]-\omega\eta(0)\,k_\mathrm{p}^{\frac{1}{2}}\,N^{\frac{1}{2}}[-4\mathrm{i}c_1N^{\frac{1}{5}}+(1+\mathrm{i})\,(-2c_4+2c_1(c_3-\tfrac{1}{8}c_1^3))\,N^{\frac{4}{5}}]=0,\end{aligned} \qquad \text{(A 46)}$$

$$\begin{aligned}-4E_1k_\mathrm{p}&+(\mathrm{i}\omega dk_\mathrm{p}^{\frac{1}{2}}/N^{\frac{1}{10}})\,[4N^{\frac{1}{5}}(q\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}+1+c_1)+(1-\mathrm{i})\,(2c_4-2c_1(c_3-\tfrac{1}{8}c_1^3))\,N^{\frac{4}{5}}]\\&-(\mathrm{i}\omega\eta'''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{5}{2}})\,[-4\mathrm{i}c_1N^{\frac{1}{5}}+(1+\mathrm{i})\,(2c_1(c_3-\tfrac{1}{8}c_1^3)-2c_4)\,N^{\frac{4}{5}}]\\&-(\mathrm{i}\omega\eta''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{3}{2}})[4+4c_1(c_3-\tfrac{1}{8}c_1^3)\,N^{\frac{4}{5}}]\\&+(\mathrm{i}\omega\eta'(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{1}{2}})\,[-4\mathrm{i}c_3N^{\frac{3}{5}}+(1-\mathrm{i})\,(2c_4-c_1^3+\tfrac{1}{4}c_1^4)\,N^{\frac{4}{5}}]\\&+\mathrm{i}\omega\eta(0)\,k_\mathrm{p}^{\frac{1}{2}}\,N^{\frac{1}{2}}\,[2c_1^2\,N^{\frac{2}{5}}]=0,\end{aligned} \qquad \text{(A 47)}$$

$$\begin{aligned}(\mathrm{i}\omega dk_\mathrm{p}^{\frac{1}{2}}/N^{\frac{1}{10}})&\,[4(q\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}+1+c_1+\tfrac{1}{2}c_1^2)\,N^{\frac{2}{5}}]-(\omega\eta'''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{5}{2}})\,[2c_1^2\,N^{\frac{2}{5}}]\\&+(\omega\eta'''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{3}{2}})\,[-4\mathrm{i}c_1N^{\frac{1}{5}}+(1+\mathrm{i})\,(2c_1(c_3-\tfrac{1}{8}c_1^3)-2c_4)\,N^{\frac{4}{5}}]\\&+(\omega\eta'(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{1}{2}})\,[4+4c_1(c_3-\tfrac{1}{8}c_1^3)\,N^{\frac{4}{5}}]\\&-\omega\eta(0)\,k_\mathrm{p}^{\frac{1}{2}}\,N^{\frac{1}{2}}[-4\mathrm{i}c_3N^{\frac{3}{5}}+(1-\mathrm{i})\,(2c_4-c_1^3+\tfrac{1}{4}c_1^4)\,N^{\frac{4}{5}}]=0,\end{aligned} \qquad \text{(A 48)}$$

and

$$\begin{aligned}(\mathrm{i}\omega dk_\mathrm{p}^{\frac{1}{2}}/N^{\frac{1}{10}})&\,[4(q\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}+1+c_1+c_3+\tfrac{1}{2}c_1^2)\,N^{\frac{3}{5}}+(1+\mathrm{i})\,(2c_4-2c_1c_3+2c_1^2+\tfrac{1}{4}c_1^4+2c_1^3)\,N^{\frac{4}{5}}]\\&-(\mathrm{i}\omega\eta'''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{5}{2}})\,[-4ic_3N^{\frac{3}{5}}+(1-\mathrm{i})\,(2c_4+\tfrac{1}{4}c_1^4-c_1^3)\,N^{\frac{4}{5}}]-(\mathrm{i}\omega\eta''(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{3}{2}})\,[2c_1^2N^{\frac{2}{5}}]\\&+(\mathrm{i}\omega\eta'(0)\,N^{\frac{1}{2}}/k_\mathrm{p}^{\frac{1}{2}})\,[-4\mathrm{i}c_1N^{\frac{1}{5}}+(1+\mathrm{i})\,(-2c_4+2c_1(c_3-\tfrac{1}{8}c_1^3))\,N^{\frac{4}{5}}]\\&+\mathrm{i}\omega\eta(0)\,k_\mathrm{p}^{\frac{1}{2}}\,N^{\frac{1}{2}}[4+4c_1(c_3-\tfrac{1}{8}c_1^3)\,N^{\frac{4}{5}}]=0.\end{aligned} \qquad \text{(A 49)}$$

(We note that the coefficients of the equations simplify considerably on use of the fact that $c_1^2-2c_2=0$.)

These equations are general for the unbaffled case and not specific to any given edge constraint. With the equations in this form it is a relatively simple matter to substitute for the known values of the two constants prescribed by any given edge constraint and then to solve for the corresponding values of E_0, E_1, etc. In particular, for the unbaffled plate with a free edge, we obtain

$$\left.\begin{aligned}\eta(0)&\sim-d[1-(10)^{\frac{1}{2}}\cot\tfrac{1}{5}\pi\,\mathrm{e}^{\frac{1}{8}\pi\mathrm{i}}],\\ \partial\eta(0)/\partial x_1&\sim-(\mathrm{i}dk_\mathrm{p}/N^{\frac{1}{5}})[1+\mathrm{i}(10)^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{8}\pi\mathrm{i}}],\\ E_0&\sim(\omega dk_\mathrm{p}^{\frac{1}{2}}\,10^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{8}\pi\mathrm{i}}/N^{\frac{1}{10}})[1-(1/2\sin^2\tfrac{1}{5}\pi-\tfrac{3}{5}-\tfrac{1}{5}\alpha)\,N^{\frac{4}{5}}],\\ \text{and}\qquad E_1&\sim-(\omega d\,10^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{8}\pi\mathrm{i}}\,N^{\frac{1}{10}}/k_\mathrm{p}^{\frac{1}{2}})\,[1+O(N^{\frac{4}{5}})].\end{aligned}\right\} \qquad \text{(A 50)}$$

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A.3. *The Wiener–Hopf factorization of the kernel* $M(k) = (k^2-k_0^2)^{\frac{1}{2}}-\bar{\mu}$

In this Appendix we now pursue the multiplicative decomposition of $M(k)$ into factors $M_\pm(k)$ possessing the usual analyticity properties (that is the factors $M_\pm(k)$ are regular and non-zero in $R_\pm$ respectively and have the convenient additional normalization $M_+(-k) = M_-(k)$).

An exact comparison of the incident, reflected and scattered wavepowers can only be made if the values of $|M_+(\alpha)|$ and $|M_-(-k_0\cos\theta)|$ are known.

Now

$$M(k) = \gamma-\bar{\mu} \quad \text{and} \quad \alpha = (\bar{\mu}^2+k_0^2)^{\frac{1}{2}}, \tag{A 51}$$

where γ is the ubiquitous acoustic square root function; whence,

$$M_\pm(k) = A_\pm[(k\pm\alpha)/(k\pm k_0)^{\frac{1}{2}}]\,L_\pm(\gamma), \tag{A 52}$$

with $A_- = A_+^{-1}$; the required normalization then fixes A_+:

$$A_+ = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}. \tag{A 53}$$

Since $\ln L(\gamma)$ satisfies the conditions of theorem B of Noble (1958) p. 15 (*q.v.*), the factors $L_\pm$ are given in terms of familiar Cauchy integrals:

$$L_\pm(k) = \exp\left\{\pm\frac{1}{2\pi\mathrm{i}}\int_{C_\pm} \ln\left[\frac{(t^2-k_0^2)^{\frac{1}{2}}}{(t^2-k_0^2)^{\frac{1}{2}}+\bar{\mu}}\right]\frac{\mathrm{d}t}{(t-k)}\right\}. \tag{A 54}$$

In particular

$$L_+(\alpha) = \exp I(\alpha), \tag{A 55}$$

where

$$I(\alpha) = \frac{1}{2\pi\mathrm{i}}\int_{C_+} \ln\left[\frac{(t^2-k_0^2)^{\frac{1}{2}}}{(t^2-k_0^2)^{\frac{1}{2}}+\bar{\mu}}\right]\frac{\mathrm{d}t}{(t-\alpha)}, \tag{A 56}$$

and the contour of integration runs from $-\infty$ to ∞ in the strip of analyticity indented *below* the pole at $t = \alpha$.

Referring to the discussion of Noble (1958, pp. 18–20), we see that the most useful of the procedures adopted there is to deform the contour of integration into the upper half-plane and on to the sides of the branch cut which consists of the real axis from k_0 to 0 and the positive imaginary axis. (See figure 3.) However, since $|\alpha| > k_0$, the indentations shown in the figure are superfluous and instead a residue contribution, associated with the pole at $t = \alpha$, is incurred. We find that the integrals on opposite sides of the branch line combine to give

$$I(\alpha) = \ln\tfrac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{y\,\mathrm{arccot}\,(k_0^2+y^2)^{\frac{1}{2}}/\bar{\mu}\,\mathrm{d}y}{(y^2+\alpha^2)}$$
$$+\frac{1}{\pi}\int_0^{k_0}\frac{\mathrm{arccot}\,(k_0^2-x^2)^{\frac{1}{2}}/\bar{\mu}\,\mathrm{d}x}{(\alpha-x)}-\frac{\mathrm{i}\alpha}{\pi}\int_0^\infty\frac{\mathrm{arccot}\,(k_0^2+y^2)^{\frac{1}{2}}/\bar{\mu}\,\mathrm{d}y}{(y^2+\alpha^2)}. \tag{A 57}$$

Some additional manipulation and evaluation of certain elementary integrals, which arise therefrom, yield

$$I(\alpha) = \tfrac{1}{2}\ln\left[\frac{1}{2}\left(\frac{\alpha+k_0}{\alpha-k_0}\right)^{\frac{1}{2}}\right]-\frac{\alpha}{2\pi}\int_0^{k_0^2/\bar{\mu}^2}\frac{\arctan t^{\frac{1}{2}}\,\mathrm{d}t}{(t+1)\,(k_0^2-\bar{\mu}^2t)^{\frac{1}{2}}}-\tfrac{1}{4}\mathrm{i}\pi\left[1-\frac{2\alpha}{\pi^2}\int_{k_0^2/\bar{\mu}^2}^\infty\frac{\arctan t^{\frac{1}{2}}\,\mathrm{d}t}{(t+1)\,(\bar{\mu}^2t-k_0^2)^{\frac{1}{2}}}\right]. \tag{A 58}$$

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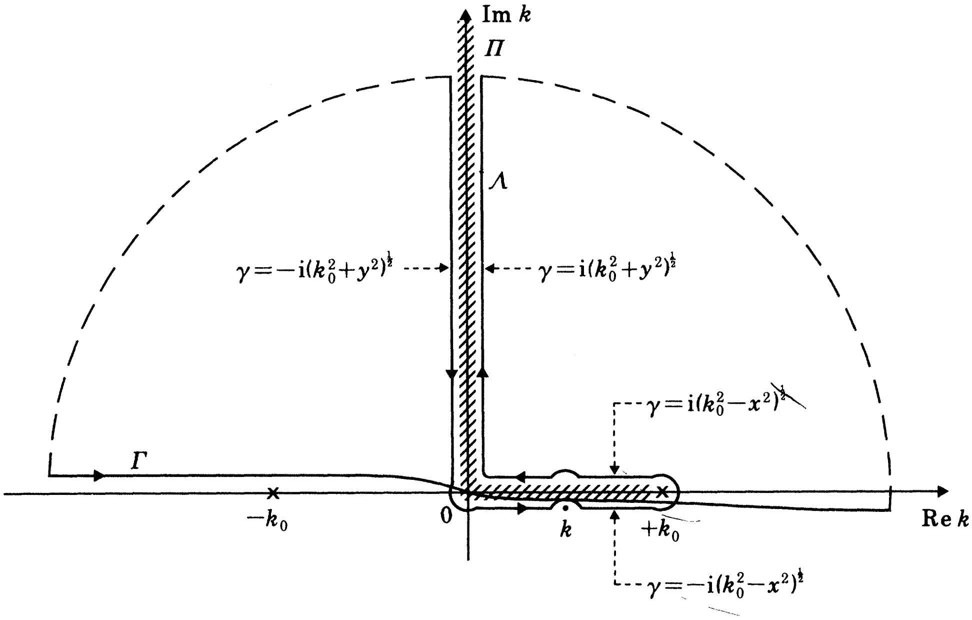

FIGURE 3. The complex plane showing the path, Λ, onto which the original contour of integration, Γ, for the Cauchy integral is deformed. Π is the L-shaped branch cut associated with the acoustic square root function.

In the limit of vanishing compressibility $\bar{\mu} = \alpha$ and then

$$I(\alpha) = \ln\sqrt{\tfrac{1}{2}} - \tfrac{1}{8}\mathrm{i}\pi. \tag{A 59}$$

When this value is substituted into (A 55) and (A 52) in turn, the value of $M_+(\alpha)$ corresponding to $k_0 = 0$ is revealed to be

$$M_+(\alpha) = (2\alpha)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{3}{8}\pi\mathrm{i}}. \tag{A 60}$$

This result may be recovered by the logarithmic differentiation of the incompressible dispersion function $|k|-\bar{\mu}$ (with $|k|$ to be interpreted as $k\operatorname{sgn}\operatorname{Re}k$). We do not include the details, which are rather mundane, here.

For arbitrary values of k_0,

$$L_+(\alpha) = \left[\frac{1}{2}\left(\frac{\alpha+k_0}{\alpha-k_0}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \exp\left[-\frac{\alpha}{2\pi}\int_0^{k_0^2/\bar{\mu}^2} \frac{\arctan t^{\frac{1}{2}}\,\mathrm{d}t}{(t+1)\,(k_0^2-\bar{\mu}^2 t)^{\frac{1}{2}}}\right] \exp\,(\mathrm{i}\Phi), \tag{A 61}$$

where

$$\Phi = -\tfrac{1}{4}\pi\left[1-\frac{2\alpha}{\pi^2}\int_{k_0^2/\bar{\mu}^2}^{\infty} \frac{\arctan t^{\frac{1}{2}}\,\mathrm{d}t}{(t+1)\,(\bar{\mu}^2 t-k_0^2)^{\frac{1}{2}}}\right]. \tag{A 62}$$

For our purposes the evaluation of Φ is not vital since its value is not needed (explicitly) in any of the energy balance calculations. Observing that

$$\int_0^{\lambda^2} \frac{\arctan t^{\frac{1}{2}}\,\mathrm{d}t}{(t+1)\,(\lambda^2-t)^{\frac{1}{2}}} = \frac{\pi}{(1+\lambda^2)^{\frac{1}{2}}}\ln\,(1+\lambda^2)^{\frac{1}{2}}, \tag{A 63}$$

and hence, on using (4.20), (A 61) and (A 52), it emerges that

$$|R| = (1+k_0^2/\bar{\mu}^2)^{-\frac{1}{2}}. \tag{A 64}$$

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The evaluation of $M_-(-k_0\cos\theta)$ is similar. We find that

$$|M_-(-k_0\cos\theta)| = (\alpha+k_0\cos\theta)^{\frac{1}{2}}. \tag{A 65}$$

Substitution of this value into the integral expression for the radiated power leads to integrands that are easily evaluated in terms of elementary functions by reference to Gradshteyn & Ryzhik (1965).

A.4. *The Wiener–Hopf factorization for the oblique incidence kernel*

The kernel which arises in §6 in connection with the oblique incidence of surface waves on the edge of a membrane is

$$K(k,\sin\theta_0) = [k^2-(k_m^2-\kappa^2\sin^2\theta_0)]\,[k^2-(k_0^2-\kappa^2\sin\theta_0)]^{\frac{1}{2}}-\mu k_m^2. \tag{A 66}$$

In the heavy fluid loading limit,

$$\mu/k_m = \sigma^3/M \quad \text{with } \sigma = O(1) \text{ and } M \ll 1.$$

Furthermore, in this limit, $\kappa \sim k_m\sigma/M^{\frac{1}{3}}$. Thus,

$$K(k,\sin\theta_0) \approx (k_m^2\sigma^3/M)\,\mathscr{K}(t,\sin\theta_0), \tag{A 67}$$

where

$$\mathscr{K}(t,\sin\theta_0) = (t^2+\sin^2\theta_0)(t^2+\sin^2\theta_0)^{\frac{1}{2}}-1, \tag{A 68}$$

with $k=\kappa t$.

To remove the zeros of $\mathscr{K}(t,\sin\theta_0)$ it is convenient to introduce the function

$$L(t) = [(t^2+\sin^2\theta_0)\,(t^2+\sin^2\theta_0)^{\frac{1}{2}}-1]\,(t^2-\cos^2\theta_0)^{-1}, \tag{A 69}$$

and then

$$\mathscr{K}_+(t,\sin\theta_0) = A_+(t+\cos\theta_0)\,L_+(t), \tag{A 70}$$

where A_+ is an arbitrary constant.

Although $L(t)$ increases as $|t|$ tends to infinity in the strip of analyticity, it is still possible to apply the decomposition theorems to $L(t)$ directly provided that the integrals which occur are understood in the sense

$$\lim_{A\to\infty}\int_{\mathrm{i}a-A}^{\mathrm{i}a+A}\frac{\ln L(k)\,\mathrm{d}k}{(k-t)}$$

(cf. Levine & Schwinger 1948, §v and Appendix B). Bearing this in mind,

$$L_+(t) = \exp F_+(t), \tag{A 71}$$

with

$$F_+(t) = \lim_{R\to\infty}\left\{\frac{1}{2\pi\mathrm{i}}\int_{-R+\mathrm{i}\epsilon}^{R-\mathrm{i}\epsilon}\ln\left[\frac{(k^2+\sin^2\theta_0)\,(k^2+\sin^2\theta_0)^{\frac{1}{2}}-1}{k^2-\cos^2\theta_0}\right]\frac{\mathrm{d}k}{(k-t)}\right\}. \tag{A 72}$$

First we require the value of $F_+(\mathrm{i}\sin\theta_0)$. Deform the countour of integration on to the sides of the vertical branch cut stretching from $\mathrm{i}\sin\theta_0$ to infinity in the upper half-plane. The result is that

$$F_+(\mathrm{i}\sin\theta_0) = \frac{\sin\theta}{2\pi}\int_0^\infty\frac{\arctan t^{\frac{3}{2}}\,\mathrm{d}t}{t(t+\sin^2\theta_0)^{\frac{1}{2}}} \tag{A 73}$$

and we have been unable to evaluate this integral.

A similar evaluation of $F_+(\cos\theta_0)$ can be achieved; in this case we also pick up a residue

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contribution when the pole at $t = \cos\theta_0$ is crossed in the contour deformation, but the procedure is straightforward and leads to

$$F_+(\cos\theta_0) = \ln\tfrac{3}{2} - \frac{\mathrm{i}\sin\theta_0\cos\theta_0}{\pi}\lim_{R\to\infty}\int_1^{R/\sin\theta_0}\frac{\arctan\,[\sin^3\theta_0(u^2-1)^{\frac{3}{2}}]\,\mathrm{d}u}{(u^2\sin^2\theta_0+\cos^2\theta_0)}$$
$$+\frac{1}{\pi}\lim_{R\to\infty}\left\{\int_1^{R/\sin\theta_0}\frac{u\sin^2\theta_0\arctan\,[\sin^3\theta_0(u^2-1)^{\frac{3}{2}}]\,\mathrm{d}u}{(u^2\sin^2\theta_0+\cos^2\theta_0)}-\int_1^R\frac{\pi}{2u}\,\mathrm{d}u\right\}. \quad (\mathrm{A}\ 74)$$

We shall not go into details, but simply state that prolonged manipulation of the integrals is fruitful and ultimately yields

$$F_+(\cos\theta_0) = \ln\sqrt{\tfrac{3}{2}} - \frac{\mathrm{i}\cos\theta_0}{2\pi}\int_0^\infty\frac{\arctan t^{\frac{3}{2}}\,\mathrm{d}t}{(t+1)\,(t+\sin^2\theta_0)^{\frac{1}{2}}}. \quad (\mathrm{A}\ 75)$$

Moreover, using (A 75) and (A 73) in conjunction with (A 72) and (A 70) supplies the required expressions for $\mathscr{K}_+(\cos\theta_0)$ and $\mathscr{K}_+(\mathrm{i}\sin\theta_0)$, namely

$$K_+(\kappa\cos\theta_0) = \frac{k_{\mathrm{m}}^{\frac{3}{2}}}{N^{\frac{1}{2}}}\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\,6^{\frac{1}{2}}\cos\theta_0\exp\left\{-\frac{\mathrm{i}\cos\theta_0}{2\pi}\int_0^\infty\frac{\arctan t^{\frac{3}{2}}\,\mathrm{d}t}{(t+1)\,(t+\sin^2\theta_0)^{\frac{1}{2}}}\right\} \quad (\mathrm{A}\ 76)$$

and

$$K_+(\mathrm{i}\kappa\sin\theta_0) = \frac{k_{\mathrm{m}}^{\frac{3}{2}}}{N^{\frac{1}{2}}}\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\,\mathrm{e}^{\mathrm{i}\theta_0}\exp\left\{\frac{\sin\theta_0}{2\pi}\int_0^\infty\frac{\arctan t^{\frac{3}{2}}\,\mathrm{d}t}{t(t+\sin^2\theta_0)^{\frac{1}{2}}}\right\}, \quad (\mathrm{A}\ 77)$$

where we have replaced $\sigma^{\frac{3}{2}}/M^{\frac{1}{2}}$ by $1/N^{\frac{1}{2}}$ as appropriate in the low-frequency limit.

These two expressions and the angle of phase shift, Θ, may be written in a more convenient form by introducing the integral $I(\lambda)$ defined by

$$I(\lambda) = \frac{1}{2\pi}\int_0^\infty\frac{\arctan u^{\frac{3}{2}}\,\mathrm{d}u}{(u+\lambda)\,(u+\sin^2\theta_0)^{\frac{1}{2}}}. \quad (\mathrm{A}\ 78)$$